On compact Riemann surfaces and hypermaps of genus $p + 1$
where $p$ is prime

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Compact Riemann surfaces
Definition & moduli

**Definition** A **Riemann surface** is a complex analytic manifold of dimension one.

Let $\mathcal{M}_g$ denote the **moduli space** of compact Riemann surfaces of genus $g \geq 2$.

- $\mathcal{M}_g$ has a structure of complex analytic space,
- its dimension is $3g - 3$, and
- if $g \geq 4$ then its singular locus is

$$\text{Sing}(\mathcal{M}_g) = \{[S] : S \text{ has non-trivial automorphisms}\}$$
Equivalences

Algebraic curves & Fuchsian groups

Assume the genus to be at least two.

**Theorem** There is an *equivalence* between:

- compact Riemann surfaces,
- (complex projective smooth) algebraic curves,
- orbit spaces of the upper-half plane

\[ \mathbb{H} := \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \]

by the action of (co-compact) **Fuchsian groups**: discrete subgroups of

\[ \text{Aut}(\mathbb{H}) \cong \text{PSL}(2, \mathbb{R}) \]
Fuchsian groups
Signature & triangle Fuchsian groups

Let $\Delta$ be a Fuchsian group such that $\mathbb{H}/\Delta$ is compact.

**Definition** The **signature** of $\Delta$ is the tuple

$$\sigma(\Delta) = (h; m_1, \ldots, m_l)$$

where:

- $h$ is the **genus of the quotient** $\mathbb{H}/\Delta$ and
- $m_1, \ldots, m_l$ are the **branch indices** in the universal canonical projection

$$\mathbb{H} \to \mathbb{H}/\Delta.$$

We will be particularly interested in **triangle** Fuchsian groups: those with signature

$$(0; a, b, c) \rightarrow \text{we simply write } (a, b, c)$$
Uniformization and group actions

Riemann’s existence theorem

**Riemann’s existence Theorem** Let $S \cong \mathbb{H}/\Gamma$ be a compact Riemann surface of genus $g \geq 2$. A finite group $G$ acts on $S$ if and only if there is a Fuchsian group $\Delta$ and a group epimorphism (ske) $\theta : \Delta \to G$ such that $\ker(\theta) = \Gamma$.

The group $G$ is said to act on $S$ with signature $\sigma(\Delta)$ and the Riemann-Hurwitz formula is satisfied

$$2(g - 1) = |G|(2h - 2 + \sum_{j=1}^{l}(1 - 1/m_j))$$

where $\sigma(\Delta) = (h; m_1, \ldots, m_l)$. 
Example

Consider a Fuchsian group of signature \((2,2,4,4)\)

\[
\Delta = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 : \gamma_1^2 = \gamma_2^2 = \gamma_3^4 = \gamma_4^2 = \prod_{i=1}^{4} \gamma_i = 1 \rangle
\]

and the \text{ ske }

\[
\theta : \Delta \rightarrow G_{5,4} = \langle a, b : a^5 = b^4 = 1, bab^{-1} = a^r \rangle
\]

\((r \text{ is a primitive } 4\text{-th root of } 1 \text{ in } \mathbb{F}_5)\) given by

\[
(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \mapsto (b^2, ab^2, ab, b^{-1}),
\]

It then follows that

\[
S = \mathbb{H}/\ker(\theta)
\]

is a compact Riemann surface and

\[
G_{5,4} \text{ acts on } S \text{ with signature } (2,2,4,4)
\]
Example

The Riemann-Hurwitz formula reads

\[ 2g - 2 = 4 \cdot 5(2 \cdot 0 - 2 + 2(1 - 1/2) + 2(1 - 1/4)) \iff g = 6 \]

Conclusion There exists a Riemann surface of genus \( g = 6 \) admitting an action of \( G_{5,4} \) with signature \( (2, 2, 4, 4) \)

A one-dimensional family of compact Riemann surfaces in the singular locus \( \text{Sing}(\mathcal{M}_6) \).
Admissible sequences

Definition & general problem

Let $a, b$ be rational numbers. The sequence

$$ag + b \quad \text{for} \quad g = 2, 3, \ldots$$

is called admissible if for infinitely many values of $g$ there is a Riemann surface of genus $g$ with a group of automorphisms of order $ag + b$.

We denote by

$$\mathcal{A}_{a,b} \subset \text{Sing}(\mathcal{M}_g)$$

the set consisting of the respective surfaces.

**General problem**

Describe $\mathcal{A}_{a,b}$
Admissible sequences

Classical examples of admissible and well-studied sequences:

\[ 84g - 84, \ 8g + 8, \ 4g + 2 \]

General Questions The following questions arise naturally.

- How many compact Riemann surfaces lie in \( A_{a,b} \)?
- Which are the possible groups of automorphisms of the members of \( A_{a,b} \)?
- Which are the possible signatures arising by the action of the previous groups?
- How many different actions appear, once the group and signature are fixed?

Notice that, in general, the set \( A_{a,b} \) need not be a family.
Example: the admissible sequence $4g$

Bujalance, Costa and Izquierdo\(^1\)

- How many compact Riemann surfaces lie in $\mathcal{A}_{4,0}$? complex dimension 1
- Which are the possible groups of automorphisms of the members of $\mathcal{A}_{4,0}$? the dihedral group only
- Which are the possible signatures arising by the action of the previous groups? $(0;2,2,2,2g)$ only.
- How many different actions appear, once the group and signature are fixed? only one

Surprisingly $\mathcal{A}_{4,0}$ is a family (one stratum $=$ the actions are all equivalent) without any additional condition on $g$.

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The case $A_{a, -a}$

The possibilities for Riemann surfaces $S$ of genus $g$ and their automorphism groups depend heavily on the factorisation of

$$\chi(S) = 2 - 2g$$

The simplest case of admissible sequence to consider is

$$ag - a$$
on the assumption that $q := g - 1$ is prime.

This problem was first considered by Belolipetsky and Jones.\(^2\)

They restrict to the case $a \geq 7$ and $q$ sufficiently large (to avoid sporadic cases) and proved that $A_{a, -a}$ “splits” into three infinite series of quasiplatonic surfaces.

The case $A_{a,-a}$

The subcase $a = 4$

A group of automorphisms $G$ of a surface of genus $g$ is large if

$$|G| > 4g - 4$$

In this case, the surface is either quasiplatonic or belongs to a one-dimensional family such that the signature of the action is

$$(0;2,2,2,n)$$ for $n \geq 3$ or $$(0;2,2,3,n)$$ for $3 \leq n \leq 5$.

The case $4g - 4$ is therefore the “maximal non-large” case. These surfaces were recently considered\(^3\). Indeed, $A_{4,-4}$ consists of:

- a two-dimensional family for each $g$, and
- a one-dimensional family (if $g \equiv 2 \mod 4$).

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The case $\mathcal{A}_{a,-a}$

The subcases $a = 3, 5, 6$

The previous results were recently extended\(^4\) to the cases

\[ 3(g - 1), 5(g - 1) \text{ and } 6(g - 1). \]

- As in the case $a = 4$, for $a = 3, 6$ appear positive dimensional families of surfaces.
- The surfaces found for $a = 5$ agree with the ones with $a = 10$ obtained earlier.
- The ones obtained for $a \geq 7$ appear as special points in these families.

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The classification is now complete

**Theorem 1**

Let $S$ be a compact Riemann surface of genus $g = q + 1$ for some prime $q \geq 7$.

There is a subgroup $G \leq \text{Aut}(S)$ of order $|G| = a(g - 1) = aq$ for some integer $a \geq 3$ if and only if one of the following holds.

Notation: let $r$ be a primitive $n$-th root of unity in $\mathbb{F}_q$. Write:

$$G_{q,n} := \langle a, b : a^q = b^n = 1, bab^{-1} = a^r \rangle = C_q \rtimes_n C_n$$
<table>
<thead>
<tr>
<th>case</th>
<th>$a$</th>
<th>signature</th>
<th>group</th>
<th>$q \equiv$</th>
<th>surfaces</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>12</td>
<td>(2,6,6)</td>
<td>$G_{q,6} \times C_2$</td>
<td>1(3)</td>
<td>$S_1, \tilde{S}_1$ *</td>
</tr>
<tr>
<td>(ii)</td>
<td>10</td>
<td>(2,5,10)</td>
<td>$G_{q,10}$</td>
<td>1(5)</td>
<td>$S_2, \tilde{S}_2, S'_2, \tilde{S}'_2$</td>
</tr>
<tr>
<td>(iii)</td>
<td>8</td>
<td>(2,8,8)</td>
<td>$G_{q,8}$</td>
<td>1(8)</td>
<td>$S_3, \tilde{S}_3$</td>
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<td>(iv)</td>
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</tr>
<tr>
<td>(vi)</td>
<td>5</td>
<td>(5,5,5)</td>
<td>$G_{q,5}$</td>
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</tr>
<tr>
<td>(vii)</td>
<td>4</td>
<td>(2,2,4,4)</td>
<td>$G_{q,4}$</td>
<td>1(4)</td>
<td>$S_1, \tilde{S}_1$</td>
</tr>
<tr>
<td>(viii)</td>
<td>4</td>
<td>(25)</td>
<td>$D_{2q}$</td>
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<td>$S_2, \tilde{S}_2$</td>
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<tr>
<td>(ix)</td>
<td>3</td>
<td>(3,3,3,3)</td>
<td>$G_{q,3}$</td>
<td>1(3)</td>
<td>$S_1, \tilde{S}_1$</td>
</tr>
<tr>
<td>(x)</td>
<td>84</td>
<td>(2,3,7)</td>
<td>$\text{PSL}(2,13)$</td>
<td>13</td>
<td>$Y_1, Y_2, Y_3$</td>
</tr>
<tr>
<td>(xi)</td>
<td>48</td>
<td>(2,3,8)</td>
<td>$\text{PGL}(2,7)$</td>
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<td>$X_1, X_2$</td>
</tr>
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<td>(xii)</td>
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</tr>
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</table>
Remarks

- The actions are explicitly given in terms of surface-kernel epimorphisms.
- The table also gives the **full** automorphism groups.
- The cases $q = 2, 3, 5$ and $a = 1, 2$ lead to less uniform behavior (but they are well-known).

**Corollary.** There is no a compact Riemann surface of genus $g = q + 1$ ($q$ prime) with exactly

\[ 3(g - 1) \text{ or } 5(g - 1) \]

automorphisms!

- The families $\mathcal{C}_1, \mathcal{C}_1'$ and $\mathcal{C}_2$ are **equisymmetric**, namely, there is only one class of topological action.
\( t = 241 \)

in \( \text{Sing}(M_{242}) \)
Belyi pairs

**Definition** A Riemann surface $S$ is called a **Belyi surface** if

$$\exists \beta : S \to \mathbb{P}^1$$

is holomorphic with three critical values.

The pair $(S, \beta)$ is called a **Belyi pair**.

Amongst Belyi pairs, the **regular** ones are those for which

$$\beta : S \to S/G \cong \mathbb{P}^1$$

is given by the action of $G \leq \text{Aut}(S)$.

**Equivalently**:

- $S$ is quasiplatonic (rigid in the moduli space)
- $S$ is uniformised by a finite index normal subgroup of a triangle Fuchsian group $\Delta(a, b, c)$
Dessin d’enfants

Definition A dessin d’enfant is an embedding of a connected, bipartite graph

\[ G \rightarrow X \]

on an oriented compact topological surface \( X \) such that the components of \( X - G \) are homeomorphic to open discs.

The dessin d’enfant is called regular if its automorphism group acts transitively on their edges.

Roughly speaking, there is a correspondence between

- (regular) dessin d’enfants,
- (regular) Belyi pairs, and
- algebraic curves defined over number fields.

Grothendieck, Belyi, Shabat, Singerman, Jones, Wolfart,...
Dessin d’enfants $=$ hypermaps

A special kind of dessin d’enfants are those that have “black” vertices all with degree 2:

- **clean** dessin d’enfants,
- **maps** (forget the colours!)
- **platonic surfaces**, namely, uniformised by finite index subgroups of Fuchsian groups of signature $\Delta(2,b,c)$

Thus, for the **regular** case:

\[
\text{regular hypermap} \equiv \text{regular dessin d’enfant} \equiv \text{regular Belyi pair} \equiv \text{quasiplatonic surface}
\]
The classification of dessins d’enfant or hypermaps

Theorem 2
The orientably regular maps/hypermaps (or, equivalently, regular dessin d’enfants on Riemann surfaces) of genus

\[ g = q + 1 \] for some prime \( q \geq 7 \)

with orientation-preserving automorphism group \( G \) of order divisible by \( q \), are given in the following table.

- up to duality/triality, permuting the roles of vertices, edges and faces.
- \( N \) is the number of orientably regular maps/hypermaps supported by the surfaces.
The classification of dessins d’enfant or maps/hypermaps

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<tr>
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Final comments

- The small cases are identified in Conder’s list.
- A similar classification for the non-orientable case of characteristic $-q$ is derived (the associated orientable double cover are the cases (x), (xi) and (xii)). This is related to earlier work of Conder, Širáň and Tucker\(^5\).
- We also provide isogeny decomposition of the associated Jacobian varieties with group action. For instance, the surfaces in case (i) decompose as

$$JS \sim E \times E' \times JX^6$$

where $E, E'$ are elliptic curves and $X$ is a quotient of $S$.

Further details


Thanks! - Hvala!