Regularity and finite element approximation for two-dimensional elliptic equations with line Dirac sources

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Outline

1. Background

2. The regularity in Sobolev space and weighted Sobolev space

3. Finite element algorithm and optimal error estimates

4. Numerical illustrations
The main problem

We are interested in the regularity and the finite element method for solving the elliptic boundary value problem [H. Li, et al. 2021]

\[-\Delta u = \delta_\gamma \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.\]  

(1)

Here,

- $\Omega \subset \mathbb{R}^2$ be a polygonal domain;
- $\gamma$ be a line segment strictly contained in $\Omega$;
- $\delta_\gamma$ is the line Dirac measure on $\gamma$, namely,

\[
\langle \delta_\gamma, v \rangle = \int_\gamma v(s)ds, \quad \forall \ v \in L^2(\gamma).
\]

Applications: monophasic flows in porous media, tissue perfusion or drug delivery by a network of blood vessels.
Literature work on FEM

\( \gamma \) degenerates to a point:
- \( L^2 \) (or \( H^\epsilon \) with small \( \epsilon \)) convergence [Babuška 1972, Scott 1973, 1976, Casas 1985];
- Convergence rate with graded meshes [Apel 2011];
- Optimal error estimates away from singular points in 2D and 3D [Koppl 2014].

\( \gamma \) is a curve:
- Assuming regularity in a weighted Sobolev space, optimal error estimate in 3D [DAngelo 2008, DAngelo 2012];
- Regularity later proved in [Ariche 2016];
- \( \gamma \) is a closed loop in 2D, element immersed interface methods [Heltai. 2019, 2020].
Challenges and Objectives

The main challenges:

- Limited regularity because of the singular source term: singular points, and singular line.
- The convergence of the finite method is slow.

The main objectives:

- Derive the regularity in a Sobolev space and weighted Sobolev space when $\gamma$ is a line segment.
- Propose the finite element algorithm.
- Obtain the optimal error estimates.
Lemma
Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Then $\delta_\gamma \in H^{-\frac{1}{2}-\epsilon}(\Omega)$ for any $\epsilon > 0$.

Lemma
Given $\epsilon > 0$, the solution of equation (1) satisfies $u \in H^{\frac{3}{2}-\epsilon}(\Omega) \cap H^1_0(\Omega)$.

Corollary
The solution $u$ of equation (1) is Hölder continuous $u \in C^{0,1/2-\epsilon}(\Omega)$ for any small $\epsilon > 0$. In particular, we have $u \in C^0(\Omega)$.

Figure: Domain $\Omega$ containing a line fracture $\gamma$. 

Regularity estimates in weighted spaces

- WLOG, \( \gamma = \{(x,0), \ 0 < x < 1\} \) with the endpoints \( Q_1 = (0,0) \) and \( Q_2 = (1,0) \).

- \( \mathcal{V} \): Singular set, which is the collection of \( Q_1, Q_2 \), and all the vertices of \( \Omega \).

The transmission problem Consider the equation

\[
\begin{cases}
-\Delta w = 0 & \text{in } \Omega \setminus \gamma, \\
 w^+_y = w^-_y - 1 & \text{on } \gamma, \\
 w^+ = w^- & \text{on } \gamma, \\
 w = 0 & \text{on } \partial\Omega,
\end{cases}
\]

where \( w_y = \partial_y w \). Here, for a function \( \nu \), \( \nu^\pm := \lim_{\epsilon \to 0} \nu(x,y \pm \epsilon) \).

It is clear that equation (2) has a unique weak solution

\[ w \in H^1(\Omega \setminus \gamma) \cap \{w |_{\partial\Omega} = 0\}. \]
Domain decomposition:

(i) the interior region $R_1 = \Omega^+ \cup \Omega^-$ away from the set $\mathcal{V}$;

(ii) the region $R_2 = B_1 \cup B_2$ consisting of the neighborhoods of the endpoints of $\gamma$;

(iii) $R_3 = \Omega \setminus (\overline{R}_1 \cup \overline{R}_2)$ is the region close to the boundary $\partial \Omega$ [Grisvard, 1985].
Weighted Sobolev spaces

**Definition**

Let $r_i(x, Q_i)$ be the distance from $x$ to $Q_i \in \mathcal{V}$ and let

$$
\rho(x) = \prod_{Q_i \in \mathcal{V}} r_i(x, Q_i).
$$

(3)

For $a \in \mathbb{R}$, $m \geq 0$, and $G \subset \Omega$, the weighted Sobolev space

$$
\mathcal{K}^m_a(G) := \{v, \rho^{\alpha}|-a \partial^\alpha v \in L^2(G), \forall |\alpha| \leq m\},
$$

where the multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_{\geq 0}^2$, $|\alpha| = \alpha_1 + \alpha_2$, and $\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{y_2}^{\alpha_2}$. The $\mathcal{K}^m_a(G)$ norm for $v$ is defined by

$$
\|v\|_{\mathcal{K}^m_a(G)} = \left( \sum_{|\alpha| \leq m} \iint_G |\rho^{\alpha}|^{-a} \partial^\alpha v|^2 \, dx \, dy \right)^{\frac{1}{2}}.
$$

**In the neighborhood** $B_i$:

$$
\mathcal{K}^m_a(B_i) = \{v, r_i^{\alpha}|^{-a} \partial^\alpha v \in L^2(B_i), \forall |\alpha| \leq m\}.
$$
Function space at the singular points

- Away from the set $V$, the weighted space $K_a^m$ is equivalent to the Sobolev space $H^m$;

Define
- $\chi_i \in C_0^\infty(B_i)$ that satisfies
  $$\chi_i = \begin{cases} 
  1 & \text{in } B(Q_i, d), \\
  0 & \text{on } \partial B_i.
  \end{cases}$$
- the linear span of these two functions
  $$W = \text{span}\{\chi_i\}, \quad i = 1, 2,$$ (4)
Regularity in $R_1$ and $R_2$

\[
\begin{cases}
-\Delta w = 0 & \text{in } \Omega \setminus \gamma, \\
w_+^+ = w_-^- - 1 & \text{on } \gamma, \\
w_+^- = w_-^+ & \text{on } \gamma, \\
w = 0 & \text{on } \partial \Omega,
\end{cases}
\]

Lemma

The solution of equation (2) is smooth in either $\Omega^+$ or in $\Omega^-$. Namely, for any $m \geq 1$, $w \in H^{m+1}(\Omega^+)$ and $w \in H^{m+1}(\Omega^-)$.

Theorem

Let $B_{d,i} := B(Q_i, d) \subset B_i$, $i = 1, 2$. Then, in $B_{d,i}$, the solution $w$ of equation (2) admits a decomposition $w = w_{\text{reg}} + w_s$, where $w_s \in W$ and $w_{\text{reg}} \in K_{a+1}^{m+1}(B_{d,i} \setminus \gamma)$ for $0 < a < 1$ and $m \geq 1$. Moreover, we have

\[
\|w_{\text{reg}}\|_{K_{a+1}^{m+1}(B_{d,i} \setminus \gamma)} + \|w_s\|_{L^\infty(B_i)} \leq C. \tag{5}
\]
Theorem

The solution $u$ of equation (1) is smooth in the region away from the set $\mathcal{V}$, namely, for $m \geq 1$, $u \in H^{m+1}(\Omega^+)$ and $u \in H^{m+1}(\Omega^-)$.

In the neighborhood of each endpoint of $\gamma$, $u$ admits a decomposition

$$u = u_{\text{reg}} + u_s, \quad u_s \in \mathcal{W},$$

such that for any $m \geq 1$ and $0 < a < 1$,

$$\|u_{\text{reg}}\|_{K_{a+1}^m(B_d,i \setminus \gamma)} + \|u_s\|_{L^\infty(B_i)} \leq C.$$

In the region $R_3$ away from $\gamma$ and close to the boundary, $u \in K_{a+1}^{m+1}(R_3)$ for $m \geq 1$ and $0 < a < \frac{\pi}{\omega}$, where $\omega$ is the largest interior angle among all the vertices of the domain $\Omega$. 
Finite element algorithm

- \( \mathcal{T} = \{ T_i \} \) be a triangulation of \( \Omega \) with triangles
- \( S(\mathcal{T}, m) = \{ v \in C^0(\Omega) \cap H^1_0(\Omega) : v|_T \in P_m(T), \ \forall \ T \in \mathcal{T} \} \), where \( P_m(T) \) is polynomials with degree no more than \( m \).
- the finite element solution \( u_h \in S(\mathcal{T}, m) \) of equation (1) by

\[
\int_{\Omega} \nabla u_h \cdot \nabla v_h \, dx = \int_{\gamma} v_h \, dx, \quad \forall \ v_h \in S(\mathcal{T}, m). \tag{6}
\]

Error estimate on quasi-uniform meshes

- the mesh \( \mathcal{T} \) consists of quasi-uniform triangles with size \( h \)
- \( u \in H^{3/2-\epsilon}(\Omega) \), the standard error estimate [Ciarlet, 1974] yields only a sup-optimal convergence rate

\[
\| u - u_h \|_{H^1(\Omega)} \leq C h^{1-\epsilon}, \quad \text{for } \epsilon > 0. \tag{7}
\]
Algorithm (Graded refinements)

Let $Q$ be also a vertex in a triangulation $\mathcal{T}$. Let $pq$ be an edge in
the triangulation $\mathcal{T}$ with $p$ and $q$ as the endpoints.

1. (Neither $p$ nor $q$ coincides with $Q$.) We choose $r$ as the midpoint ($|pr| = |qr|$).

2. ($p$ coincides with $Q$.) We choose $r$ such that $|pr| = \kappa |pq|$, where $\kappa \in (0, 0.5)$ is a parameter that will be specified later. See Figure 3 for example.

Figure: The new node on an edge $pq$ (left – right): $p \neq Q$ and $q \neq Q$
(midpoint); $p = Q$ ($|pr| = \kappa |pq|$, $\kappa < 0.5$).
Graded refinements (Con’t)

Figure: Refinement of a triangle $\triangle x_0 x_1 x_2$. First row: (left – right): the initial triangle and the midpoint refinement; second row: two consecutive graded refinements toward $x_0 = Q$, ($\kappa < 0.5$).
Theorem
Recall $\kappa_{Q} = 2^{-\frac{m}{a}}$ for the graded mesh on $T(0)$, $m \geq 1$ and $0 < a < 1$. Let $S_n$ be the finite element space associated with the graded triangulation $T_{n}$ defined in Algorithm 2. Let $u_n \in S_n$ be the finite element solution of equation (1). Then,

$$\|u - u_n\|_{H^1(\Omega)} \leq Ch^m \leq C\text{dim}(S_n)^{-\frac{m}{2}},$$

where $\text{dim}(S_n)$ is the dimension of $S_n$. 

Optimal error estimates on graded meshes
Example 1 (Union-Jack meshes and graded meshes)

Example

- square domain $\Omega = (0, 1)^2$, FEM: $P_1$ polynomials
- $\gamma = Q_1 Q_2$ has two vertices $Q_1 = (0.25, 0.5)$ and $Q_2 = (0.75, 0.5)$

Figure: Graded mesh and Union-Jack mesh. (a) and (b): the initial Union-Jack mesh and the mesh after one refinement. (c) and (d): the initial graded mesh and the mesh after one refinement, $\kappa = \kappa_{Q_1} = \kappa_{Q_2} = 0.2$. 
Table: Convergence history with mesh refinements.

<table>
<thead>
<tr>
<th>$\kappa \backslash j$</th>
<th>$j = 2$</th>
<th>$j = 3$</th>
<th>$j = 4$</th>
<th>$j = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa = 0.1$</td>
<td>0.99</td>
<td>0.94</td>
<td>0.97</td>
<td>0.99</td>
</tr>
<tr>
<td>$\kappa = 0.2$</td>
<td>0.97</td>
<td>0.99</td>
<td>0.99</td>
<td>1.00</td>
</tr>
<tr>
<td>$\kappa = 0.3$</td>
<td>0.87</td>
<td>0.96</td>
<td>0.99</td>
<td>1.00</td>
</tr>
<tr>
<td>$\kappa = 0.4$</td>
<td>0.86</td>
<td>0.91</td>
<td>0.94</td>
<td>0.98</td>
</tr>
<tr>
<td>$\kappa = 0.5$</td>
<td>0.84</td>
<td>0.87</td>
<td>0.89</td>
<td>0.91</td>
</tr>
<tr>
<td>Union-Jack</td>
<td>0.46</td>
<td>0.47</td>
<td>0.49</td>
<td>0.49</td>
</tr>
</tbody>
</table>

- Union-Jack meshes: the convergence rate shall be about 0.5.
- Graded meshes: optimal when $\kappa := \kappa Q_1 = \kappa Q_2 = 2^{-\frac{1}{a}} < 0.5$
Example 3

- triangle domain $\Omega = \Delta ABC$ with $A = (0, 0), B = (1, 0)$ and $C = (0.5, 1)$, FEM: $P_2$ polynomials
- $\gamma = Q_1 Q_2$ with $Q_1 = (0.3, 0.25), Q_2 = (0.7, 0.25)$

Figure: Quadratic finite element methods on graded meshes with the line fracture $\gamma = Q_1 Q_2$, $Q_1 = (0.3, 0.25), Q_2 = (0.7, 0.25)$. (a) the initial mesh; (b) the mesh after four refinements, $\kappa = \kappa Q_1 = \kappa Q_2 = 0.2$; (c) the numerical solution.
Example 2

Table: Convergence history of the $P_2$ elements on graded meshes.

<table>
<thead>
<tr>
<th>$\kappa \setminus j$</th>
<th>$j = 4$</th>
<th>$j = 5$</th>
<th>$j = 6$</th>
<th>$j = 7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa = 0.1$</td>
<td>1.74</td>
<td>1.86</td>
<td>1.94</td>
<td>1.97</td>
</tr>
<tr>
<td>$\kappa = 0.2$</td>
<td>1.81</td>
<td>1.88</td>
<td>1.93</td>
<td>1.97</td>
</tr>
<tr>
<td>$\kappa = 0.3$</td>
<td>1.65</td>
<td>1.68</td>
<td>1.70</td>
<td>1.71</td>
</tr>
<tr>
<td>$\kappa = 0.4$</td>
<td>1.32</td>
<td>1.32</td>
<td>1.32</td>
<td>1.32</td>
</tr>
<tr>
<td>$\kappa = 0.5$</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

- all the interior angles of $\Omega$ are less than $\frac{\pi}{2}$, the solution is in $H^3$ except for the region that contains $\gamma$.
- optimal when $\kappa := \kappa_{Q_1} = \kappa_{Q_2} = 2^{-\frac{2}{a}} < 0.25$ due to the fact $0 < a < 1$
Conclusion
▶ derived the regularity in both Sobolev space and weighted Sobolev space
▶ Proposed a finite element algorithm.
▶ obtained the optimal error estimates.

Future plan
▶ $\gamma$ is a plane in a 3D domain.
▶ Consider similar source term in biharmonic problem [H. Li, P. Yin, Z. Zhang].
H. Li, X. Wan, P. Yin, L. Zhao.

H. Li, P. Yin, Z. Zhang.