# Invariant measures for a stochastic nonlinear and damped 2D Schrödinger equation

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# Deterministic 2D NLS equation

$$\begin{cases} \mathrm{d}u(t) = -\left[\mathrm{i}Au(t) + \mathrm{i}|u(t)|^{\alpha-1}u(t)\right]\,\mathrm{d}t, \qquad t>0, \qquad \alpha>1, \\ u(0)=u_0, \end{cases} \tag{1}$$

on 2D compact Riemannian manifold $(M, g)$ without boundary and on relatively compact smooth domains in $\mathbb{R}^2$ with Dirichlet/Neumann boundary conditions.
$A$ is the realization of a Laplacian type operator on these domains.
Spaces where to look for solutions: $H = L^2(\mathcal{O})$, $V = D(A^{\frac{1}{2}})$.

- The nonlinear Schrödinger equation occurs as a basic model in hydrodinamics, plasma physics, nonlinear optics, molecular biology. It describes the propagation of waves media with both nonlinear and dispersive responses.

- Conserved quantities:

$$\mathcal{M}(u) = \|u\|_H^2, \qquad u \in H,$$

$$\mathcal{E}(u) = \frac{1}{2}\|\nabla u\|_H^2 + \frac{1}{\alpha+1}\|u\|_{L^{\alpha+1}}^{\alpha+1}, \qquad u \in V.$$

# Stochastic and damped 2D NLS equation

$$\begin{cases} \mathrm{d}u(t) = -\left[\mathrm{i}Au(t) + \mathrm{i}|u(t)|^{\alpha-1}u(t) + \beta u(t)\right] dt \\ \qquad\qquad -\mathrm{i}Bu(t) \circ \mathrm{d}W(t) - \mathrm{i}G(u(t))\,\mathrm{d}\mathbf{W}(t), \qquad t > 0, \qquad \alpha > 1, \\ u(0) = u_0. \end{cases} \tag{2}$$

- The addition of the noise term destroys the conservation of mass and energy.
- Adding a damping term one hopes in some balance between the dissipation term and the forcing term to reach an equilibrium.
- We look for invariant measures of equation (2). Existing results:
  - Kim and Ekren-Kukavica-Ziane on $\mathbb{R}^d$, $d \geq 1$,
  - Debussche-Odasso on 1D bounded domain (prove also uniqueness).
- As a preliminary result we need to ensure existence and uniqueness of the solution.

# Existence of an invariant measure

Assumptions on G:

$$\exists\, C_1, \tilde C_1 > 0 \ s.t. \ \ \|G(u)\|_{\gamma(Y_2,H)} \le C_1 + \tilde C_1 \|u\|_H \qquad \forall u \in H.$$

$$\exists\, C_2, \tilde C_2 > 0 \ s.t. \ \ \|G(u)\|_{\gamma(Y_2,V)} \le C_2 + \tilde C_2 \|u\|_V \qquad \forall u \in V,$$

$$\exists\, C_3, \tilde C_3 > 0 \ s.t. \ \ \|G(u)\|_{\gamma(Y_2,L^{\alpha+1})} \le C_3 + \tilde C_3 \|u\|_{L^{\alpha+1}} \qquad \forall u \in L^{\alpha+1}.$$

Assumptions on $B$:

$$B \in \mathscr{L}(H, \gamma(Y_1, H)), \quad B \in \mathscr{L}(V, \gamma(Y_1, V)), \quad B \in \mathscr{L}(L^{\alpha+1}, \gamma(Y_1, L^{\alpha+1})).$$

Linear damping: does not provide any regularization.

**Theorem (Z. Brzeźniak, B. Ferrario, M.Z.)**

*Let $u_0 \in V$. There exists at least one invariant measure for equation (2), with support contained in $V$, provided*

$$\beta > \max\left( \tilde C_1^2 + \tilde C_2^2 + \|B\|^2_{\mathscr{L}(V,\gamma(Y_1,V))}, \frac{\alpha+1}{2} \|B\|^2_{\mathscr{L}(L^{\alpha+1},\gamma(Y_1,L^{\alpha+1}))} + \alpha \tilde C_3^2 \right).$$

In the pure additive noise the condition becomes $\beta > 0$: in line with results in $\mathbb{R}^d$ (Kim, Ekren-Kukavica-Ziane).

# Existence of a martingale solution

It is based on the compactness of the embedding $V \subset L^p$ and the Hamiltonian structure of the equation: since these ingredients are independent of the underlying geometry, the proof works in a more general setting.

- Introduce a Galerkin approximation sequence,
- prove the tightness of the law of this sequence in the space $C([0, T, V^*) \cap L^{\alpha+1}(0, T; L^{\alpha+1}) \cap C_w([0, T], V)$,
- conclude by a tightness argument and the Martingale Representation Theorem.

The same apriori estimates needed for the proof of the existence of a martingale solution are used to get the existence of an invariant measure.

# Pathwise uniqueness

- We gain some regularity on the solution by means of the Strichartz estimates of Blair-Smith-Sogge 2008: for every $x \in D(A^{\frac{2}{3p}})$,

$$\|e^{-itA}x\|_{L^p([0,T];L^q)} \lesssim_T \|x\|_{D(A^{\frac{2}{3p}})}, \quad \frac{2}{q} + \frac{2}{p} = 1, \ (p,q) \neq (2,\infty).$$

- This regularity is enough to control the non linear term.
- We get pathwise uniqueness in $H$, by means of a Schmalfuss argument.
- Existence of a martingale solution and pathwise uniqueness ensure the existence of a strong solution.

# Existence of an invariant measure: idea of the proof

- The proof relies on the Maslowski-Seidler "version" of the Krylov Bogoliugov Theorem.
- Given the family $\{P_t\}_{t\geq 0}$

$$P_t\phi(u_0) = \mathbb{E}[\phi(u(t;u_0))],\ \ t\geq 0, \phi\in B_b(V),$$

to ensure the existence of at least one invariant measure for (2), we need to show that

1. $\{P_t\}_{t\geq 0}$ is a Markov semigroup sequential weak Feller in $V$, i.e. $P_t : SC_b(V_w)\to SC_b(V_w), \forall t>0$,
2. for any $\varepsilon>0$ there exists $R_\varepsilon>0$ such that

$$\sup_{T\geq 1}\frac{1}{T}\int_0^T \mathbb{P}\left(\|u(t;0)\|_V > R_\varepsilon\right)dt < \varepsilon.$$

# Uniqueness of the invariant measure in the purely multiplicative case

**Theorem**

*Assume* $C_1 = 0$*, that is* $\|G(u)\|_{\gamma(Y_1,H)} \leq \tilde{C}_1 \|u\|_H, \ \forall u \in H$*. If*

$$\beta > \frac{1}{2}\tilde{C}_1^2,$$

*then there exists a unique invariant measure for equation* (2) *given by* $\pi = \delta_0$*.*

# Thank you!