Pointwise convergence for the Schrödinger equation with orthonormal initial data

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The Hartree equation

$$\begin{aligned} i\partial_t u &= (-\Delta_x + w * |u|^2)u \\ u(0,x) &= f(x) \end{aligned}$$

on $\mathbb{R} \times \mathbb{R}^n$ models dynamics of a Bose–Einstein condensate in $\mathbb{R}^n$ in which all quantum particles occupy the same state $u(t,x)$

Well-posedness fairly well understood (Strichartz estimates,...)

Taking the interaction potential w as δ_0 yields cubic NLS
In this case, Compaan–Lucà–Staffilani (2019) showed

$$\lim_{t\to 0} u(t,x) = f(x) \qquad \text{for a.e. } x$$

whenever $f \in H^s(\mathbb{R}^n)$ and $s > \max(\frac{n}{2(n+1)}, \frac{n-2}{2})$

§1 Carleson's problem

When $w = 0$ the pointwise convergence problem is known as Carleson's problem i.e. identifying minimal s such that

$$\lim_{t \to 0} e^{it\Delta} f(x) = f(x) \qquad \text{for a.e. } x \in \mathbb{R}^n$$

whenever $f \in H^s(\mathbb{R}^n)$

Critical exponent is $s = s(n) = \frac{n}{2(n+1)}$

E.g. $s(1) = \frac{1}{4}$ and $\lim_{n \to \infty} s(n) = \frac{1}{2}$

Carleson, Dahlberg–Kenig ($n = 1$), Bourgain, Du–Guth–Li ($n = 2$), Bourgain, Du–Zhang ($n \geq 3$)

§2 Fermions

Unlike bosons, fermions cannot occupy the same state (Pauli exclusion principle)

Described by N orthonormal functions $u_1, \dots, u_N$ in $L^2(\mathbb{R}^n)$

Dynamics modelled by system of N Hartree equations

$$\begin{aligned} i\partial_t u_j &= (-\Delta_x + w * \rho) u_j \\ u_j(0,x) &= f_j(x) \qquad\qquad (j = 1, \dots, N) \end{aligned}$$

where

$$\rho(t,x) = \sum_{k=1}^{N} |u_k(t,x)|^2$$

is the total density of particles at time t, and $f_1, \dots, f_N$ are orthonormal

Well-posedness when $N = \infty$?

Density matrices

Define operator $\gamma(t) = \sum_{j=1}^{N} \Pi_{u_j(t)}$ by

$$\gamma(t)f := \sum_{j=1}^{N} \langle f, u_j(t) \rangle u_j(t)$$

System of N Hartree equations is equivalent to

$$i\partial_t \gamma = [-\Delta + w * \rho_\gamma, \gamma]$$
$$\gamma(0) = \sum_{j=1}^{N} \Pi_{f_j}$$

where

$$\rho_\gamma(t,x) = K_\gamma(t,x,x)$$

and K_γ is the integral kernel of $\gamma(t)$

Quick calculation reveals $\rho_\gamma = \rho$

Density matrices

$N = \infty$ can be formulated through

$$\begin{aligned} i\partial_t \gamma &= [-\Delta + w * \rho_\gamma, \gamma] \\ \gamma(0) &= \gamma_0 \end{aligned}$$

where γ_0 is a more general (non-negative) operator

If $\gamma_0 = \sum_j \lambda_j \Pi_{f_j}$ is trace class ($\lambda \in \ell^1$), solution can be written $\gamma(t) = \sum_j \lambda_j \Pi_{u_j(t)}$ where

$$\begin{aligned} i\partial_t u_j &= (-\Delta_x + w * \rho) u_j \\ u_j(0, x) &= f_j(x) \qquad\qquad (j = 1, \ldots, N) \end{aligned}$$

and the density $\rho_\gamma = \sum_j \lambda_j |u_j|^2$ is well-defined since

$$\|\rho_{\gamma(t)}\|_1 \leq \sum_j |\lambda_j| \|u_j(t, \cdot)\|_2^2 = \|\lambda\|_{\ell^1} < \infty$$

§3 Carleson's problem for fermions

Consider the density matrix equation with $w = 0$ (von Neumann Schrödinger equation)

$$
\begin{aligned}
i\partial_t \gamma &= [-\Delta, \gamma] \\
\gamma(0) &= \gamma_0
\end{aligned}
$$

Solution is $\gamma(t) = e^{it\Delta}\gamma_0 e^{-it\Delta}$
It seems natural to pose the problem: For which γ_0 do we have

$$
\lim_{t\to 0} \rho_{\gamma(t)}(x) = \rho_{\gamma_0}(x) \qquad \text{for a.e. } x
$$

A more precise version: Given an appropriate Hilbert space $\mathcal{H}$ (such as $\dot{H}^s$ or H^s), what is the largest $\beta = \beta(s, n)$ for which

$$
\lim_{t\to 0} \rho_{\gamma(t)}(x) = \rho_{\gamma_0}(x) \qquad \text{for a.e. } x \in \mathbb{R}^n
$$

whenever γ_0 belongs to the Schatten space $\mathcal{C}^\beta(\mathcal{H})$?

Finite-rank case corresponds to classical Carleson's problem

To extend to the infinite-rank case, it suffices to prove maximal-in-time estimates of the form

$$\Big\| \sum_j \lambda_j |e^{it\Delta} f_j|^2 \Big\|_{L_x^{q/2} L_t^\infty} \lesssim \|\lambda\|_{\ell^\beta}$$

where (f_j) is orthonormal in $\mathcal{H}$

Strichartz estimates in this framework take the form

$$\Big\| \sum_j \lambda_j |e^{it\Delta} f_j|^2 \Big\|_{L_t^{q/2} L_x^{r/2}} \lesssim \|\lambda\|_{\ell^\beta}$$

where (f_j) is orthonormal in $\dot{H}^s(\mathbb{R}^n)$

Frank–Lewin–Lieb–Seiringer, Frank–Sabin, Chen–Hong–Pavlovic, B–Hong–Lee–Nakamura–Sawano, B–Lee–Nakamura,...

Certain endpoint cases open – back to this later

A result in 1D

Theorem (B–Lee–Nakamura)

The (weak-type) maximal-in-time estimate

$$\Big\| \sum_j \lambda_j |e^{it\Delta} f_j|^2 \Big\|_{L_x^{2,\infty} L_t^\infty(\mathbb{R}\times\mathbb{R})} \lesssim \|\lambda\|_{\ell^\beta}$$

holds for all (f_j) *orthonormal in* $\dot{H}^{1/4}(\mathbb{R})$ *if and only if* $\beta < 2$

Consequently

$$\lim_{t\to 0} \rho_{\gamma(t)}(x) = \rho_{\gamma_0}(x) \qquad \textit{for a.e. } x \in \mathbb{R}$$

holds whenever $\gamma_0 \in \mathcal{C}^\beta(\dot{H}^{1/4})$ *and* $\beta < 2$

A problem of Frank–Sabin

Theorem (B–Lee–Nakamura)

Let $a > 0$ and $a \neq 1$. If $\beta < 2$, the (weak-type) maximal-in-space estimate

$$\Big\| \sum_j \lambda_j |e^{it(-\Delta)^{a/2}} f_j|^2 \Big\|_{L_t^{2,\infty} L_x^\infty(\mathbb{R}\times\mathbb{R})} \lesssim \|\lambda\|_{\ell^\beta}$$

holds for all (f_j) orthonormal in $\dot{H}^{\frac{1}{2}-\frac{a}{4}}(\mathbb{R})$

When $a = 2$, Frank–Sabin obtained that $\beta = \frac{2r}{r+2}$ is optimal for $L_t^{q/2} L_x^{r/2}(\mathbb{R}\times\mathbb{R})$ whenever $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$ and $r \in [2, \infty)$
But, for $r = \infty$, only the (trivial) case $\beta = 1$ was known

In 1D, maximal-in-space imply maximal-in-time since we can switch the roles of space and time

$$x\xi + t\xi^2 = t\eta + x\sqrt{\eta}$$

via the change of variables $\eta = \xi^2$

Minor snag: orthogonality of data breaks, but this can be recouped via some symmetrisation