# Universal sequences and Euler tours in hypergraphs


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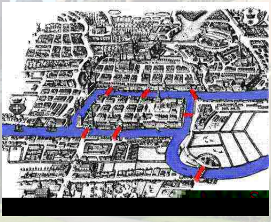

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# Universal sequences

A universal cycle for $[n]$ and $k$ is a cyclic sequence with elements in $[n]$, such that every $k$-subset of $[n]$ appears exactly once consecutively.

**Example:** 1234524135 is a universal cycle for [5] and 2.

**Conjecture (Chung, Diaconis and Graham, 1989 ($100))**

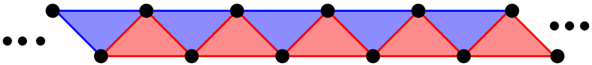
*For any $k$ and any sufficiently large $n$, universal cycles for $[n]$ and $k$ exist if and only if $k \mid \binom{n-1}{k-1}$.*

Divisibility condition is necessary:
every element is contained in $\binom{n-1}{k-1}$ many $k$-sets,
every time an element appears in the sequence, it appears in $k$ consecutive $k$-sets

# Euler circuits in hypergraphs

A tight Euler tour in a 3-uniform hypergraph $G$:

- cyclic sequence of vertices
- every 3 consecutive vertices span a hyperedge of $G$
- every hyperedge of $G$ occurs exactly once in this way



Definition naturally generalizes to $k$-uniform hypergraphs

**Conjecture (Chung, Diaconis and Graham, 1989 ($100))**

*For sufficiently large n, the complete n-vertex k-uniform hypergraph $K_n^k$ has a tight Euler tour if and only if $k \mid \binom{n-1}{k-1}$.*

Equivalent to conjecture on universal sequences

# Euler circuits: previous results

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- Curtis, Hines, Hurlbert, Moyer (2009): approximate solution

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CHHM: *At the 2004 Banff Workshop ... it was suggested.. that a modest inflationary rate should revalue the prize near* 250.04*.... Due to our proof that near-universal cycles exist, we believe that we deserve asymptotically much of the prize money, or* $(1 - o(1))(250.04)$*. Since we do not know the speed of the o(1) term, we have made a conservative estimate of* 249.99*.*

# Euler circuits

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**Theorem (Glock, Joos, Kühn, Osthus, 2020)**

*The conjecture is true.*

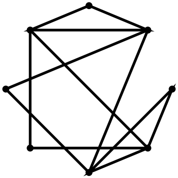
based on existence of $F$-designs (Glock, Lo, Kühn, Osthus, $17^+$)

# Detour: $F$-decompositions and $F$-designs

**Hypergraph decompositions**

*$G$ has an F-decomposition if there exist pairwise edge-disjoint copies of $F$ in $G$ which cover all (hyper)edges of $G$.*

If $G = K_n$ and $F = K_3$ this is a Steiner triple system of order $n$.



If $G$ is complete, then this is an $F$-design

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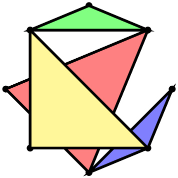

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# Detour: $F$-decompositions and $F$-designs

$F$-designs exist for any $F$:

**Theorem (Glock, Kühn, Lo, Osthus $2017^+$)**

*Suppose $F$ is a $k$-uniform hypergraph and suppose that the complete $n$-vertex $k$-uniform hypergraph $K_n^{(k)}$ is $F$-divisible, where $n \gg |V(F)|$. Then $K_n^{(k)}$ has an $F$-decomposition.*

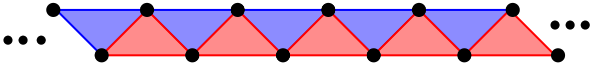
- clique case (ie $F$ is complete) is due to Keevash
- graph case $k = 2$ is due to Wilson
- can replace $K_n^{(k)}$ by 'almost complete' host hypergraph $G$
- proof is combinatorial and is based on iterative absorption

# Proof strategy I: Random walk

**Theorem (Glock, Joos, Kühn, Osthus, 2020)**

*For sufficiently large $n$, $K_n^k$ has a tight Euler tour if and only if $k \mid \binom{n-1}{k-1}$.*

Consider random walk on the vertex set $[n]$ which does not 'revisit' $k$-sets:



**Lemma (Universality lemma)**

*Almost surely, after $n^{k-1}(\log n)^2$ steps, the random walk $W$ has traversed every $k-1$-element ordered set.*

So $K_n^k - W$ is still almost complete!

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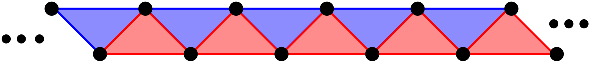


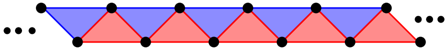
**Lemma (Universality lemma)**

*Almost surely, after $n^{k-1}(\log n)^2$ steps, the random walk $W$ has traversed every $k-1$-element ordered set.*

So $K_n^k - W$ is still almost complete!
This still holds if we turn $W$ into a closed walk by adding a few more steps

# Proof strategy II: Completion via $F$-designs

Let $F$ be a tight cycle (ie no repeated vertices) of length $2k$



We have:

- $G = K_n^k - W$ is $F$-divisible!
- $G$ is also almost complete

$\Rightarrow$ by GKLO-theorem, $G$ has an $F$-decomposition

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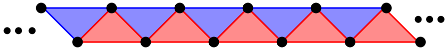

We have:

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Let $C_1, \ldots, C_m$ be the cycles of the $F$-decomposition.
For each $i$, let $T_i$ be an ordered $k-1$-set traversed by $C_i$.

Insert $C_i$ into $W$ when $W$ visits $T_i$.
Since $W$ visits every ordered $k-1$-set, we eventually insert every cycle in this way.

$\Rightarrow$ the resulting walk is an Euler tour

# Open problems: Euler circuits

Recall main result:

**Theorem (Glock, Joos, Kühn, Osthus, 2020)**

*For sufficiently large $n$, $K_n^k$ has a tight Euler tour if and only if $k \mid \binom{n-1}{k-1}$.*

What about Euler tours in noncomplete hypergraphs?

- Decision problem is NP-complete
- our result holds for almost complete hypergraphs

**Conjecture**

*Every $k$-uniform hypergraph $G$ with $\delta_{k-1}(G) \geq (\frac{k-1}{k} + o(1))n$ has a tight Euler tour if all vertex degrees are divisible by $k$.*

True for $k = 3$ (Piga and Sanhueza-Matamala $2021^+$)

# Open problems: Oberwolfach problem

The Oberwolfach problem has a solution for all sufficiently large $n$.

**Theorem (Glock, Joos, Kim, Kühn, Osthus, $18^+$)**

*$\exists n_0$ such that for all odd $n \geq n_0$ and any cycle factor $F$ on $n$ vertices, $K_n$ has an $F$-decomposition.*

What about Oberwolfach decompositions of graphs of large degree?

**Conjecture**

*$\exists n_0$ such that for $n \geq n_0$ and any cycle factor $F$ on $n$ vertices, any even-regular graph $G$ with $\delta(G) \geq 3n/4$ has an $F$-decomposition.*