Duality for noncommutative frames

8ECM

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Based on joint work with Karin Cvetko-Vah and Lieven Le Bruyn.

Karin Cvetko-Vah, Jens Hemelaer and Lieven Le Bruyn, Duality for noncommutative frames, Topology and its Applications (2020), arXiv:1911.12625.

Skew lattices

A skew lattice is a set S with two associative, idempotent binary operations $\wedge$ and $\vee$, such that

$$x \wedge (x \vee y) = x = x \vee (x \wedge y)$$

and

$$(x \wedge y) \vee y = y = (x \vee y) \wedge y$$

for all $x, y \in S$.

If $\wedge$ and $\vee$ are commutative, then S is a lattice.

Strong distributivity

We say that S is strongly distributive if

$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$$

and

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Leech showed that this property is equivalent to S satisfying the three conditions below.

- symmetric: $x \vee y = y \vee x \Leftrightarrow x \wedge y = y \wedge x$;
- distributive: $x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$ and $x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x)$;
- normal: $x \wedge y \wedge z \wedge x = x \wedge z \wedge y \wedge x$.

Structure of skew lattices

We say that $a, b \in S$ are in the same $\mathcal{D}$-class, or $a \, \mathcal{D} \, b$, if

$$a \wedge b \wedge a = a \quad \text{and} \quad b \wedge a \wedge b = b.$$

The $\mathcal{D}$-class of a is denoted by $[a]$. The natural partial order on S is

$$a \leq b \iff a \wedge b = b \wedge a = a.$$

Theorem (Leech's First Decomposition Theorem)

The equivalence relation $\mathcal{D}$ is a congruence and the set of $\mathcal{D}$-classes $S/\mathcal{D}$ is the maximal lattice image.

We call $S/\mathcal{D}$ the commutative shadow of S.

Structure of skew lattices

The skew lattice S is left handed if

$$x \wedge y \wedge x = x \wedge y$$

and right handed if

$$x \wedge y \wedge x = y \wedge x$$

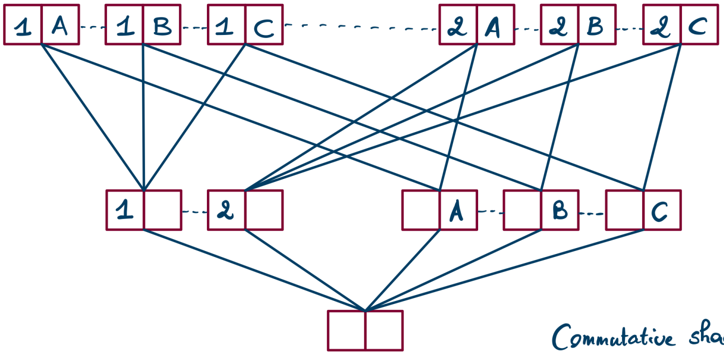
for all $x, y \in S$.

Theorem (Leech's Second Decomposition Theorem)

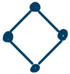
A skew lattice can be written as the pullback of a left handed skew lattice and a right handed skew lattice over their common commutative shadow.

So we will only look at left handed skew lattices.

Example



Commutative shadow:



——— natural partial order

- - - - - - same D-class

Restriction

In a normal skew lattice (a fortiori in a strongly distributive one), the subset

$$a{\downarrow} = \{b \in S : b \leq a\}$$

is a lattice. It intersects every $\mathcal{D}$-class $v \leq [a]$ in precisely one element.

We will denote this element by $a|_v$ and we will call it the restriction of a to v.

Noncommutative frames

A subset $\{x_i : i \in I\} \subset S$ is called a commuting subset if

$$x_i \wedge x_j = x_j \wedge x_i \quad \text{and} \quad x_i \vee x_j = x_j \vee x_i$$

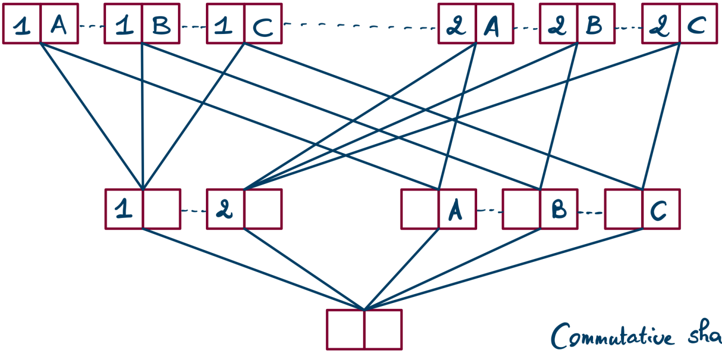
for all $i, j \in I$.

A noncommutative frame is a strongly distributive skew lattice such that every commuting subset $\{x_i : i \in I\}$ has a supremum $\bigvee_{i \in I} x_i$, and moreover

$$(\bigvee_i x_i) \wedge y = \bigvee_i (x_i \wedge y) \quad \text{and} \quad y \wedge (\bigvee_i x_i) = \bigvee_i (y \wedge x_i)$$

for a commuting subset $\{x_i : i \in I\} \subseteq S$ and $y \in S$.

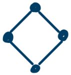
Example



natural partial order

same $\mathcal{D}$-class

Commutative shadow:



Example

Take non-empty sets Q and R. Consider the set $\mathcal{P}(Q, R)$ of partial maps $Q \to R$. We define:

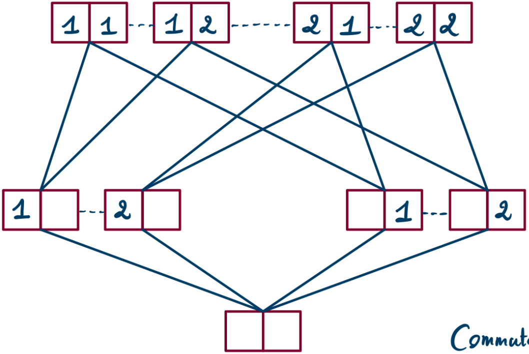
$$f \wedge g = f|_{\mathrm{dom}(f) \cap \mathrm{dom}(g)} \qquad \text{(restrict)}$$

$$f \vee g = g \cup f|_{\mathrm{dom}(f) - \mathrm{dom}(g)} \qquad \text{(overwrite)}$$

for $f, g \in \mathcal{P}(Q, R)$. Under these operations, $\mathcal{P}(Q, R)$ is a strongly distributive skew lattice (Leech, 1992) and even a (left handed) noncommutative frame.

- $f \, \mathcal{D} \, g \;\Leftrightarrow\; \mathrm{dom}(f) = \mathrm{dom}(g)$;
- $f \leq g \;\Leftrightarrow\; f = g|_{\mathrm{dom}(f) \cap \mathrm{dom}(g)}$;
- $\mathcal{P}(Q, R)/\mathcal{D} = \mathcal{P}(Q)$.

Example



$Q = \{1, 2\}$

$R = \{1, 2\}$

Commutative shadow:

——— natural partial order

- - - - - - same $\mathcal{D}$-class

The front topology

Let X be a topological space. The front topology is the topology on X generated by the open and closed sets.

Example

The front topology on $\mathbb{R}$ (or any Hausdorff space) is the discrete topology.

Notation: we write X_f for X with the front topology.

Spatial noncommutative frames

Let X be a topological space and let $\mathcal{E}$ be a sheaf on X_f such that $\mathcal{E}(X_f) \neq \varnothing$. Let A be the set consisting of the pairs (U, s) with $U \subseteq X$ open and $s \in \mathcal{E}(U)$.

We define:

$$(U, s) \wedge (V, t) = (U \cap V, s|_{U \cap V})$$
$$(U, s) \vee (V, t) = (U \cup V, s|_{U - V} \cup t).$$

The commutative shadow is the frame $\mathcal{O}(X)$ of open sets of X.

The operations can still be thought of as restrict and overwrite. Noncommutative frames of this form will be called spatial.

Are all nc frames spatial?

No.

Take a (commutative) frame A that is not of the form $\mathcal{O}(X)$ for some topological space X. Then A is also not spatial as noncommutative frame.

But even if $A/\mathcal{D} = \mathcal{O}(X)$ for a topological space X, there are counterexamples.

Are all nc frames spatial?

Consider the pairs (U, s) with $U \subseteq \mathbb{R}$ open and $s : U \to \{0, 1\}$ an arbitrary function.

We say that $(U, s) \sim (V, t)$ if and only if $U = V$ and s and t agree in each point, with at most countably many exceptions.

The set of equivalence classes A then becomes a strongly distributive skew lattice with $\wedge$ and $\vee$ defined by restrict and overwrite as before.

To show that A is a noncommutative frame, we need the property that $\mathbb{R}$ is hereditarily Lindelöf (every open covering of an open set in $\mathbb{R}$ has a countable subcovering).

Are all nc frames spatial?

How can we show that this noncommutative frame is not spatial?

Suppose that A is spatial, corresponding to a sheaf $\mathcal{E}$ on $\mathbb{R}_f$.

Then for an open set $U \subseteq \mathbb{R}$ and two elements (U, s) and (U, t) with $s \neq t$, we can find an $x \in \mathbb{R}$ such that $s(x) \neq t(x)$.

As a result, we can find a homomorphism:

with $\pi(V, r) = 0$ if $x \notin V$, $\pi(V, r) = a$ if $x \in V$ and $r(x) = s(x)$ and $\pi(V, r) = b$ if $x \in V$ and $r(x) \neq s(x)$.

Are all nc frames spatial?

But this is not possible in our example!

Suppose that (U, s) and (U, t) are two pairs in our example. For any $x \in \mathbb{R}$ we have that

$$s|_{U-\{x\}} = t|_{U-\{x\}} \;\Rightarrow\; s = t.$$

So $\pi(U, s) = \pi(U, s \vee t|_{U-\{x\}}) = \pi(U, t)$.

So where do they come from?

For strongly distributive skew lattices, already very strong results (2013) by Bauer, Cvetko-Vah, Gehrke, Van Gool and Kudryavtseva:

Strongly distributive left handed skew lattices all arise from pairs $(X, \mathcal{E})$ where X is a local Priestley space and $\mathcal{E}$ is a sheaf on X.

This is stated as an equivalence of categories.

How to relate this to the front topology? If a strongly distributive lh skew lattice A has a *bounded* distributive lattice as commutative shadow, with spectrum X, then the patch topology is the topology on X generated by compact opens and their complements. Then A corresponds to $(X, \mathcal{E})$ with $\mathcal{E}$ a sheaf on X_p.

So where do they come from?

This gives an idea on how to generalize the equivalence to noncommutative frames. Instead of taking the topology generated by open and closed sets, we allow formal complements of open sets.

We start with a noncommutative frame A with commutative shadow $L = A/\mathcal{D}$. There is a universal frame $N(L)$ such that every element of L becomes complemented in $N(L)$.

This frame $N(L)$ is rarely spatial, even if L is!
If L is the frame of opens of a locale Y, then $N(L)$ is the frame of opens of the dissolution locale Y_d.

So where do they come from?

It turns out that every left handed noncommutative frame arises from a pair $(Y, \mathcal{F})$, for Y a locale (the commutative shadow) and $\mathcal{F}$ a sheaf on the dissolution locale Y_d, with $\mathcal{F}(Y_d) \neq \varnothing$.

Thank you for listening!