On the existence of large set of partitioned incomplete Latin squares

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Joint work with Dongliang Li, Li Wang, and Haitao Cao

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Outline

- **Part I: LSPILS**$^+(1^q u^1)$
  - Definitions
  - Construction
  - Main result

- **Part II: OLSPILS**
  - Definitions
  - Constructions
  - Main results
  - Problems for further research

Part I: LSPILS$^+(1^q u^1)$

Definition 1

A **latin square** of *order* n is an $n \times n$ array L defined on an n-set X such that each symbol of X occurs exactly once in each row and exactly once in each column.

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Definition 2

A **partitioned incomplete latin square** PILS$(n; a_1, a_2, ..., a_k)$ is an $n \times n$ array L defined on X with a partition $A_1, A_2, \ldots, A_k$ (called *groups*), which satisfies the following properties:
(1) $A_i \cap A_j = \emptyset$ for $1 \leq i < j \leq k$, $|A_i| = a_i$, and $a_1 + ... + a_k = n$;

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A PILS is denoted by PILS$(a_1^{s_1} a_2^{s_2} ... a_t^{s_t})$ if there are s_i groups of size a_i, $i \in \{1, 2, ..., t\}$.

Examples

	0	1	2
0		2	1
1	2		0
2	1	0	

PILS(1^3)

	0	1	2	x	y
0		x	y	2	1
1	y		x	0	2
2	x	y		1	0
x	1	2	0		
y	2	0	1		

PILS$(1^3 2^1)$

	0	1	2	3	4	5
0			4	5	2	3
1			5	4	3	2
2	4	5			0	1
3	5	4			1	0
4	2	3	0	1		
5	3	2	1	0		

PILS(2^3)

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Two PILSs L and M are **disjoint** if $L(i,j) \neq M(i,j)$ for each non-empty cell (i,j).

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2	x	y		1	0
x	1	2	0		
y	2	0	1		

L

	0	1	2	x	y
0		y	x	1	2
1	x		y	2	0
2	y	x		0	1
x	2	0	1		
y	1	2	0		

M

Definition of LSPILS$^+(g^n u^1)$

Definition

- A *large set of partitioned incomplete latin squares* of type $g^n u^1$, denoted by LSPILS$(g^n u^1)$, is a set of mutually disjoint $g(n-1)$ PILS$(g^n u^1)$s.

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2	x	y		1	0
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0		y	x	1	2
1	x		y	2	0
2	y	x		0	1
x	2	0	1		
y	1	2	0		

	0	1	2
0		2	1
1	2		0
2	1	0	

Background

Theorem [Lei, JCD, 1997]

There exists an LGDD(g^n) if and only if $n(n-1)g^2 \equiv 0 \pmod 6$, $(n-1)g \equiv 0 \pmod 2$, and $(g,n) \neq (1,7)$.

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There exists a golf design of order n if and only if $n \equiv 1 \pmod 2$ and $n \neq 5$.

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There exists an LSPILS$^+(2^n 4^1)$ if and only if $n \equiv 0 \pmod 3$, except possibly for $n \in \{30, 48, 144\}$.

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Theorem [Shen, Li, and Cao, JCD, 2021]

There exists an LSPILS$^+(g^n(2g)^1)$ for all $g \geq 1$ and $n \geq 3$ except possibly for $n \equiv 2, 10 \pmod{12}$ and $n \geq 14$.

Definition of *QDM

Definition 1

Let F_q be a finite field of q elements, and let r be a primitive element of F_q. A *quasi-difference matrix* $Q = (q_{ij})$, denoted by QDM$(q+u, u)$, is a $3 \times (q+2u)$ array defined on F_q satisfying that
(1) each cell of Q is empty or contains an element of F_q;
(2) each row contains exactly u empty entries (usually denoted by -), and each column contains at most one empty entry;
(3) for each $1 \leq i < j \leq 3$, the multiset $\triangle_{ij} = \{q_{il} - q_{jl} : 1 \leq l \leq q+2u$, with q_{il} and q_{jl} not empty $\} = F_q$.

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Definition 2

A *QDM$(q+u,u)$ is a QDM$(q+u,u)$ satisfying that $(0,0,0)^T$ is a column, and for any two columns C and D with no empty entries, $D \neq r^i C$ holds for any $0 \le i \le q-2$. Let $Q = (q_{il})$ be a *QDM$(q+u,u)$. If a and b are integers, $q_{1l} = 0, q_{2l} = j$, and $q_{3l} = a + bj$, then *QDM$(q+u,u)$ is denoted by $Q = (a+bj)$.

PILS and *QDM

Example. *QDM$(7+2,2)$.

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & - & - & 0 & 0 \\ 0 & 5 & - & 6 & 3 & - & 4 & 2 & 1 & 1 & 2 \end{pmatrix}$$

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$$\Downarrow$$

	0	1	2	3	4	5	6	x	y
0		5	x	6	3	y	4	2	1
1									
2									
3									
4									
5									
6									
x	1								
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2	y	6		0	x	1	5	4	3
3	6	y	0		1	x	2	5	4
4	3	0	y	1		2	x	6	5
5	x	4	1	y	2		3	0	6
6	4	x	5	2	y	3		1	0
x	1	2	3	4	5	6	0		
y	2	3	4	5	6	0	1		

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1	5		6	x	0	4	y	3	2
2	y	6		0	x	1	5	4	3
3	6	y	0		1	x	2	5	4
4	3	0	y	1		2	x	6	5
5	x	4	1	y	2		3	0	6
6	4	x	5	2	y	3		1	0
x	1	2	3	4	5	6	0		
y	2	3	4	5	6	0	1		

$\Downarrow$

	0	1	2	3	4	5	6	x	y
0		y	4	1	5	2	x	6	3
1	x		y	5	2	6	3	0	4
2	4	x		y	6	3	0	1	5
3	1	5	x		y	0	4	2	6
4	5	2	6	x		y	1	3	0
5	2	6	3	0	x		y	4	1
6	y	3	0	4	1	x		5	2
x	3	4	5	6	0	1	2		
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$$Q = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&x&y \\ 0&1&2&3&4&5&6&x&y&0&0 \\ 0&5&x&6&3&y&4&2&1&1&2 \end{pmatrix} \Rightarrow 3Q = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&x&y \\ 0&3&6&2&5&1&4&x&y&0&0 \\ 0&1&x&4&2&y&5&6&3&3&6 \end{pmatrix} \Rightarrow 3^2Q \Rightarrow \dots$$

$\Downarrow$

	0	1	2	3	4	5	6	x	y
0		5	x	6	3	y	4	2	1
1	5		6	x	0	4	y	3	2
2	y	6		0	x	1	5	4	3
3	6	y	0		1	x	2	5	4
4	3	0	y	1		2	x	6	5
5	x	4	1	y	2		3	0	6
6	4	x	5	2	y	3		1	0
x	1	2	3	4	5	6	0		
y	2	3	4	5	6	0	1		

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	0	1	2	3	4	5	6	x	y
0		y	4	1	5	2	x	6	3
1	x		y	5	2	6	3	0	4
2	4	x		y	6	3	0	1	5
3	1	5	x		y	0	4	2	6
4	5	2	6	x		y	1	3	0
5	2	6	3	0	x		y	4	1
6	y	3	0	4	1	x		5	2
x	3	4	5	6	0	1	2		
y	6	0	1	2	3	4	5		

0	1	5
0	1	y
0	1	6
0	1	3
0	1	x
0	1	2
0	1	4

Main result of LSPILS from *QDM

lemma

There exists a *QDM$(q+u,u)$ for any prime power $q=p^m$, $q\geq 3$, and $2\leq u\leq q-1$.

$$Q=\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & - & - \\ 0 & \cdots & j & \cdots & j_0 & j_1 & - & - & 0 & 0 \\ 0 & & a+bj & & - & - & a & j_1 & a & a+(b-1)j_0 \end{pmatrix}, \quad \begin{array}{l} j_0=-ab^{-1} \\ j_1=a(1-b)^{-1} \\ j\in GF(q)^*\setminus\{j_0,j_1\} \end{array}$$

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Theorem

There exists an LSPILS$^+(1^q u^1)$ for any prime power $q=p^m$, $q\geq 3$, and $2\leq u\leq q-1$.

Main results of LSPILS

Definition

Let $n = \Pi_{i=1}^{t} p_i^{e_i}$, where p_i are distinct primes and e_i are positive integers. Then n is called a **regular integer** if $p_i^{e_i} \geq 4$ holds for all $1 \leq i \leq t$.

Theorem [Shen and Cao, submitted to DM]

There exists an LSPILS$^+(g^{qn}(gu)^1)$ for all positive integers g, u and regular integer n, where q is a prime power, and $q \geq u + 1 \geq 4$.

Part II: OLSPILS

Definition 1

Define $I_k = \{1, 2, ..., k\}$. Two PILSs both defined on X with groups A_i for $i \in I_k$ are *orthogonal* if upon superimposition all ordered pairs in $(X \times X) \setminus \cup_{i=1}^{k}(A_i \times A_i)$ result. A Latin square is *self-orthogonal* if it is orthogonal to its transpose.

Definition 2

Two LSPILSs, say $\{L_i : i \in I_s\}$ and $\{M_i : i \in I_s\}$, are orthogonal if each L_i and M_i are orthogonal, $i \in I_s$. An LSPILS is called *self-orthogonal* if each Latin square is self-orthogonal.

Known results of OLSPILS

Theorem [Zhu, 2006]

There exists an OLSPILS(1^q) for $q \geq 3$ a prime power.

It can always exist by taking $L_i(x, y) = a_i x + (1 - a_i)y$ and $M_i = L_{i+1}$ for $i \in I_{q-2}$, where $L_{q-1} = L_1$, $x \neq y \in GF(q)$, and $a_i \in GF(q) \setminus \{0, 1\}$.

Theorem [Huang, Liu, Ge, Ma, and Zhang, 2019]

There exist a number of new instances of a pair of mutually orthogonal partial golf designs $\{L_i : i \in I_d\}$ and $\{M_i : i \in I_d\}$, where $d \leq n - 2$. If $d = n - 2$, it is a pair of orthogonal golf designs. However, there has been no example for a pair of orthogonal golf designs so far.

Theorem [Shen, Cao, and Ji, JCD, 2020]

There exists an OLSPILS(7^5).

Constructions

i	$\dots$	0	$\dots$	$\vec{0}$	$\vec{0}$	$\vec{0}$	$\vec{\infty}$
j	$\dots$	j	$\dots$	S_0	T_0	$\vec{\infty}$	$\vec{0}$
Q_1: x_1	$\dots$	$a + bj$	$\dots$	$\vec{\infty}$	$a + bT_0$	$a + bS_1$	$a + (b-1)S_2$
Q_2: x_2	$\dots$	$c + dj$	$\dots$	$c + dS_0$	$\vec{\infty}$	$c + dT_1$	$c + (d-1)T_2$
$x_1 - x_2$	$\dots$	$a - c + (b-d)j$	$\dots$	-	-	$a - c + bS_1 - dT_1$	$a - c + (b-1)S_2 - (d-1)T_2$

Constructions

i	$\dots$	0	$\dots$	$\vec{0}$	$\vec{0}$	$\vec{0}$	$\vec{\infty}$
j	$\dots$	j	$\dots$	S_0	T_0	$\vec{\infty}$	$\vec{0}$
Q_1: x_1	$\dots$	$a+bj$	$\dots$	$\vec{\infty}$	$a+bT_0$	$a+bS_1$	$a+(b-1)S_2$
Q_2: x_2	$\dots$	$c+dj$	$\dots$	$c+dS_0$	$\vec{\infty}$	$c+dT_1$	$c+(d-1)T_2$
x_1-x_2	$\dots$	$a-c+(b-d)j$	$\dots$	-	-	$a-c+bS_1-dT_1$	$a-c+(b-1)S_2-(d-1)T_2$

lemma

Let $q=p^m$ be a prime power, where $q\geq 2u+1$ and $u\geq 2$. $Q_1=(a+bj)$, $Q_2=(c+dj)$ are two *QDM$(q+u,u)$s.
If S, T, S_i, and T_i $(i=0,1,2)$ satisfy the following conditions:
(1) $S\cap T=\emptyset$ and $|S|=|T|=u$;
(2) $\{(b-d)s_i^0 : i\in I_u^*\}\cup\{(b-d)t_i^0 : i\in I_u^*\}$
$=\{bs_i^1-dt_i^1 : i\in I_u^*\}\cup\{(b-1)s_i^2-(d-1)t_i^2 : i\in I_u^*\}$.
Then there exists an OLSPILS$(1^q u^1)$.

Example

Example. Initial triples of an OLSPILS$(1^7 3^1)$.

	i	0	0	0	0	0	0	0	0	0	a_1	a_2	a_3
	j	6	1	2	3	4	5	a_1	a_2	a_3	0	0	0
L_0:	x_1	a_1	4	a_2	a_3	6	2	1	3	5	1	5	6
M_0:	x_2	3	a_1	4	2	a_2	a_3	5	1	6	3	1	5
	$x_1 - x_2$	-	-	-	-	-	-	3	2	6	5	4	1

Constructions

lemma

Let p be a prime. There exists an OLSPILS$(1^p 3^1)$ for $p \geq 7$.

- $p = 7, 11$
- $p > 11$

Constructions

lemma

Let p be a prime. There exists an OLSPILS$(1^p 3^1)$ for $p \geq 7$.

- $p = 7, 11$
- $p > 11$

i	$\dots$	0	$\dots$	0	0	0	0	0	0	0	0	0	∞_1	∞_2	∞_3
j	$\dots$	j	$\dots$	j_1	j_2	$-b^{-1}$	$(1-b)^{-1}$	$-d^{-1}$	$(1-d)^{-1}$	∞_1	∞_2	∞_3	0	0	0
Q_1: x_1	$\dots$	$1+bj$	$\dots$	∞_1	$1+bj_2$	∞_2	∞_3	$1-bd^{-1}$	$1+b(1-d)^{-1}$	1	$1+bj_1$	$(1-b)^{-1}$	1	b^{-1}	$1+(b-1)j_1$
Q_2: x_2	$\dots$	$1+dj$	$\dots$	$1+dj_1$	∞_1	$1-db^{-1}$	$1+d(1-b)^{-1}$	∞_2	∞_3	$(1-d)^{-1}$	1	$1+dj_2$	$1+(d-1)j_2$	1	d^{-1}
x_1-x_2	$\dots$	$(b-d)j$	$\dots$	-	-	-	-	-	-	$1-(1-d)^{-1}$	bj_1	$(1-b)^{-1}-1-dj_2$	$(1-d)j_2$	$b^{-1}-1$	$1+(b-1)j_1-d^{-1}$

Constructions

lemma

Let p be a prime. There exists an OLSPILS$(1^p 3^1)$ for $p \geq 7$.

- $p = 7, 11$
- $p > 11$

	i	...	0	...	0	0	0	0	0	0	0	0	0	∞_1	∞_2	∞_3
	j	...	j	...	j_1	j_2	$-b^{-1}$	$(1-b)^{-1}$	$-d^{-1}$	$(1-d)^{-1}$	∞_1	∞_2	∞_3	0	0	0
Q_1:	x_1	...	$1+bj$	...	∞_1	$1+bj_2$	∞_2	∞_3	$1-bd^{-1}$	$1+b(1-d)^{-1}$	1	$1+bj_1$	$(1-b)^{-1}$	1	b^{-1}	$1+(b-1)j_1$
Q_2:	x_2	...	$1+dj$	...	$1+dj_1$	∞_1	$1-db^{-1}$	$1+d(1-b)^{-1}$	∞_2	∞_3	$(1-d)^{-1}$	1	$1+dj_2$	$1+(d-1)j_2$	1	d^{-1}
	x_1-x_2	...	$(b-d)j$	...	-	-	-	-	-	-	$1-(1-d)^{-1}$	bj_1	$(1-b)^{-1}-1-dj_2$	$(1-d)j_2$	$b^{-1}-1$	$1+(b-1)j_1-d^{-1}$

$$\begin{cases} (b-d)j_1 = b^{-1} - 1, \\ (b-d)j_2 = 1 - (1-d)^{-1}, \\ (b-d)(-b^{-1}) = 1 + (b-1)j_1 - d^{-1}, \\ (b-d)(1-b)^{-1} = (1-d)j_2, \\ (b-d)(-d^{-1}) = bj_1, \\ (b-d)(1-d)^{-1} = (1-b)^{-1} - 1 - dj_2. \end{cases}$$

Constructions

$$\left[\begin{array}{cc|c} b(b-d) & 0 & 1-b \\ 0 & (b-d)(1-d) & -d \\ bd(b-1) & 0 & d^2-2bd+b \\ 0 & (1-d)(1-b) & b-d \\ bd & 0 & d-b \\ 0 & (1-b)(1-d)d & b^2-2bd+d \end{array}\right]$$

$$\Rightarrow \quad \left[\begin{array}{cc|c} bd & 0 & d-b \\ 0 & (1-b)(1-d) & b-d \\ 0 & 0 & d^2-3bd+d+b^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow b^2 - 3db + d + d^2 = 0$

$\Rightarrow \triangle = 5d^2 - 4d$ is the square element of $GF(p)$.

$$\Rightarrow \begin{cases} j_1 = (d-b)(bd)^{-1}, \\ j_2 = (b-d)(1-d)^{-1}(1-b)^{-1}. \end{cases}$$

Specially, set $d = 4$ and $b = 2$.

Example

p=13																						
u=3:																						
i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	—	—	—				
j	1	2	3	4	5	6	7	8	9	10	11	12	—	—	—	0	0	0				
x1=1+2j	3	5	7	9	11	—	2	4	6	—	10	—	1	8	12	1	7	11				
x2=1+4j	5	9	—	—	8	12	3	—	11	2	6	10	4	1	7	12	1	10				
x2−x1			6	8		12		3		7		11	3	6	8	11	7	12				
u=5:																						
					j1		j3		j4		j2											
i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	—	—	—	—	—
j	1	2	3	4	5	6	7	8	9	10	11	12	—	—	—	—	—	0	0	0	0	0
x1=1+2j	3	5	7	9	—	—	2	4	6	—	—	—	1	8	10	11	12	1	6	7	11	12
x2=1+4j	5	9	—	—	8	12	—	—	—	2	6	10	4	1	11	3	7	12	2	1	10	9
x2−x1					10		1		5		9				1	5			9			10

Example

									p=13													
u=3:																						
i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	—	—	—				
j	1	2	3	4	5	6	7	8	9	10	11	12	—	—	—	0	0	0				
x1=1+2j	3	5	7	9	11	—	2	4	6	—	10	—	1	8	12	1	7	11				
x2=1+4j	5	9	—	—	8	12	3	—	11	2	6	10	4	1	7	12	1	10				
x2−x1			6	8		12		3		7		11	3	6	8	11	7	12				
u=5:																						
					j1		j3		j4		j2											
i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	—	—	—	—	—
j	1	2	3	4	5	6	7	8	9	10	11	12	—	—	—	—	—	0	0	0	0	0
x1=1+2j	3	5	7	9	—	—	2	4	6	—	—	—	1	8	10	11	12	1	6	7	11	12
x2=1+4j	5	9	—	—	8	12	—	—	—	2	6	10	4	1	11	3	7	12	2	1	10	9
x2−x1					10		1		5		9				1	5			9			10

$$\begin{cases} (d-b)j_1 = (d-1)j_3 - (b-1)j_2, \\ (d-b)j_2 = (d-1)j_4 - (b-1)j_1, \\ (d-b)j_3 = dj_4 - bj_2, \\ (d-b)j_4 = dj_3 - bj_1. \end{cases} \Rightarrow \begin{cases} j_1 = 2j_3 - j_4, \\ j_2 = -j_3 + 2j_4. \end{cases}$$

Constructions

lemma

Let $p \geq 13$ is a prime, $p \neq 19$, and $M = \{0, -1, -2^{-1}, -3^{-1}, 4^{-1}, -4^{-1}, (-2) \cdot 3^{-1}\}$. Then the set $Z_p \setminus M$ can be partitioned into $\lfloor (p-7)/4 \rfloor$ disjoint quadruples of the form $(2x-y, 2y-x, x, y)$.

Main results of OLSPILS$(1^p(2t+1)^1)$

Theorem

Let $p \geq 11$ be a prime. There exists an OLSPILS$(1^p u^1)$, where u is an odd integer and $5 \leq u \leq 3 + 2\lfloor (p-7)/4 \rfloor$.

- $t = 1, 2$:

Theorem

Let p be a prime and $p \equiv 1 \pmod 6$.
1. There exists an SOLSPILS$^+(1^p2^1)$.
2. There exists an OLSPILS(1^p4^1).

Main results of OLSPILS$(1^p(2t)^1)$

- $t = 1, 2$:

Theorem

Let p be a prime and $p \equiv 1 \pmod 6$.
1. There exists an SOLSPILS$^+(1^p 2^1)$.
2. There exists an OLSPILS$(1^p 4^1)$.

- $t > 2$?

Further research

Problem

Let p be a prime and $u \equiv 0 \pmod{2}$.
Prove the existence of an OLSPILS$(1^p u^1)$ for
(1) $u = 2, 4$ and $p \equiv 5 \pmod{6}$; and
(2) $u \geq 6$.

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Thank you for your attention!