Computing Eigenvalues of the Laplacian on Rough Domains

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joint work with
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Introduction

General question:

- Can one always compute the spectrum of the Laplacian on a domain?

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$\mathcal{O} \subset \mathbb{R}^d$ open, $-\Delta_{\mathcal{O}} =$ Dirichlet Laplacian on $\mathcal{O}$.

- Does there exist **one** sequence (Γ_N) of computer algorithms s.t.

$$\Gamma_N(-\Delta_{\mathcal{O}}) \to \sigma(-\Delta_{\mathcal{O}})$$

  for **all** $\mathcal{O}$ in a given class Ω?
- How large can Ω be?

Introduction

Definition:[1] A *computational (spectral) problem* consists of

- Class of operators Ω,
- A set Λ of *input information* (e.g. $A \mapsto \langle e_i, Ae_j \rangle$).

Definition:[1] An *Algorithm* is a map

$$\Gamma : \Omega \to [\text{closed subsets of } \mathbb{C}]$$

such that

- $\Gamma(T)$ depends only on finitely many $f \in \Lambda$,
- $\Gamma(T)$ can be computed using finitely many arithmetic operations on these $f(T)$.

[1] [Hansen(2011)], [Ben-Artzi-Colbrook-Hansen-Nevanlinna-Seidel(2020)]

Background

Recent work:

[Hansen(2011)], [Ben-Artzi-Colbrook-Hansen-Nevanlinna-Seidel(2015)]:

- Development of abstract framework for computational problems & algorithms;
- Abstract theory of computational complexity;
- Classification of computational complexity for some abstract (spectral and other) problems;

[Colbrook-Hansen(2020)], [Colbrook(2020)]:

- Classification of complexity for wider classes of spectral problems: computing spectra in $\mathbb{R}^d$, spectral measures, spectral gaps, ...

Setup

Computational problem for Laplacians on domains:

- Class of operators: $\Omega :=$ set of all bounded open subsets $\mathcal{O} \subset \mathbb{R}^2$ with
  (i) $\mathcal{O} = \overline{\mathcal{O}}^{\circ}$
  (ii) $|\partial\mathcal{O}| = 0$
  (iii) $\mathbb{R}^2 \setminus \mathcal{O}$ has finitely many connected components whose diameter is bounded below.
- Operator: Dirichlet Laplacian $-\Delta_{\mathcal{O}}$.
- Input information:

$$\Lambda = \{\mathcal{O} \mapsto \mathbb{1}_{\mathcal{O}}(x) \mid x \in \mathbb{R}^2\}.$$

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Examples:

- $\mathcal{O} =$ interior of Jordan curve (e.g. Koch Snowflake),
- $\mathcal{O} =$ filled Julia set with connected interior.

Results

Theorem 1 (R., Stepanenko, 2021): For Ω, Λ as above, there exists a sequence of algorithms $\Gamma_n : \Omega \to \mathrm{cl}(\mathbb{C})$ such that

$$\Gamma_n(\mathcal{O}) \to \sigma(\mathcal{O}) \quad \text{as} \quad n \to \infty \qquad \text{for all} \qquad \mathcal{O} \in \Omega,$$

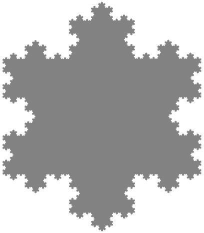
locally in Hausdorff sense.

Results

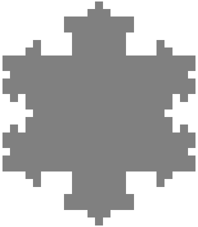
Idea of Proof:

- Approximate $\mathcal{O}$ by union $\mathcal{O}_n$ of **finitely many** small boxes,
- approximate spectrum $\sigma(\mathcal{O}_n)$ using FEM ($\rightsquigarrow$ off-the-shelf),[2]
- show that $\sigma(\mathcal{O}_n) \to \sigma(\mathcal{O})$ as $n \to \infty$ ($\rightsquigarrow$ Mosco Convergence).

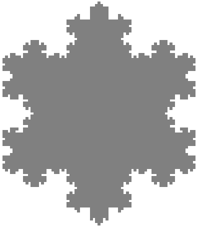
$\mathcal{O}$:



$\mathcal{O}_{30}$:



$\mathcal{O}_{90}$:



[2] [Liu-Oishi(2013)]

Results

Idea of Proof: Mosco Convergence

Theorem 2 (R., Stepanenko, 2021): Let $\mathcal{O} \in \Omega$. Suppose that $\mathcal{O}_n \subset \mathbb{R}^2$, $n \in \mathbb{N}$, is a collection of bounded, open sets such that $\partial\mathcal{O}_n$ is locally connected for all $n \in \mathbb{N}$ and such that

$$d_H(\mathcal{O}, \mathcal{O}_n) + d_H(\partial\mathcal{O}, \partial\mathcal{O}_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Then, $\mathcal{O}_n$ converges to $\mathcal{O}$ in Mosco sense as $n \to \infty$.

Results

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Proof: Need to show:

1. $H_0^1(\mathcal{O}_n) \ni u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2) \quad \Rightarrow \quad u \in H_0^1(\mathcal{O})$.
2. $u \in H_0^1(\mathcal{O}) \quad \Rightarrow \quad \exists\, u_n \in H_0^1(\mathcal{O}_n)$ with $u_n \to u$ in $H^1(\mathbb{R}^2)$.

To prove 1.:

- Take cutoff function χ_n with $\chi_n \equiv 0$ in nbhd. of $\partial\mathcal{O}$ and consider $\chi_n u_n$.
- To control $u_n \nabla\chi_n$: Need **explicit Poincaré inequality** on $\mathcal{O}_n$.

Results

Idea of Proof: Poincaré Inequality

Theorem 3 (R., Stepanenko, 2021): Let $\mathcal{O} \subset \mathbb{R}^2$ open with no arbitrarily small holes.[3] If $r > 0$ is small enough, then

$$\|u\|_{L^2(N_r(\partial\mathcal{O}))} \leq 5r\|\nabla u\|_{L^2\left(N_{2\sqrt{2}r}(\partial\mathcal{O})\right)}$$

for all $u \in H_0^1(\mathcal{O})$.

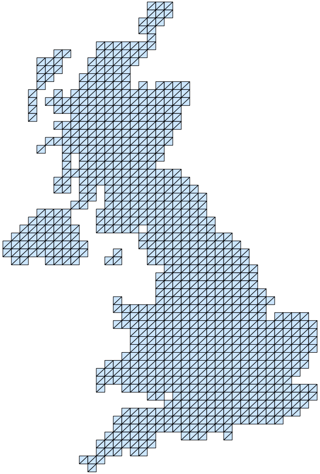
Proof: Explicit estimates in nbhd. of $\partial\mathcal{O}$, using the fact that $\mathcal{O}$ cannot have arbitrarily small holes.

[3] $\mathrm{diam}(\Gamma) > c > 0$ for all path-connected components Γ of $\partial\mathcal{O}$.

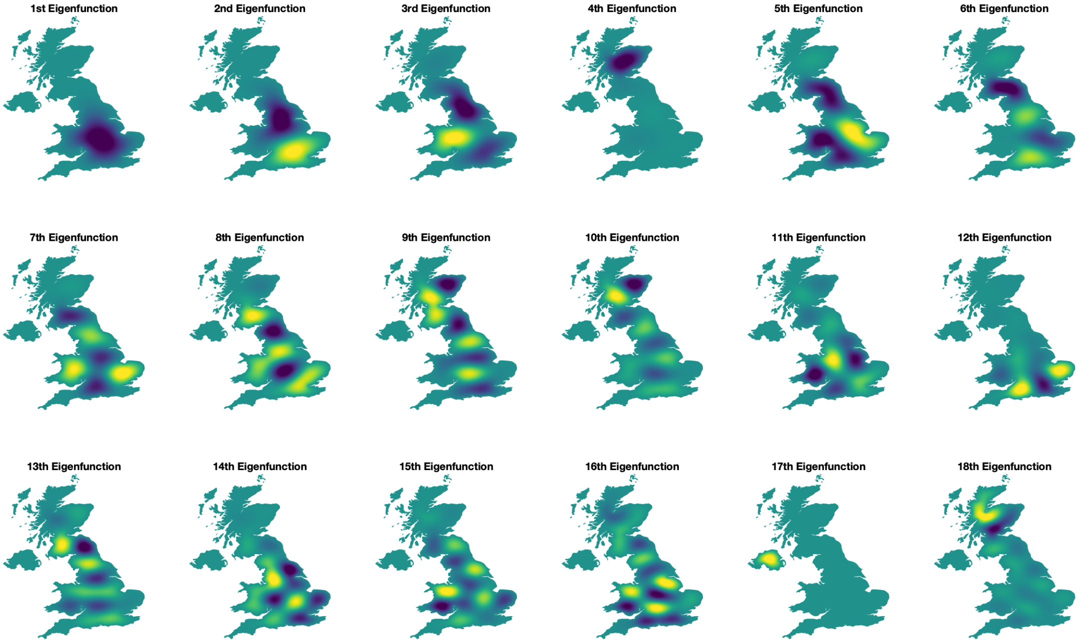
Numerical Results

Domain:

Pixelation:



Numerical Results



Thank You!