Maps on positive definite cones of C^*-algebras preserving the Wasserstein mean

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Notation.
In what follows, any C^*-algebra $\mathcal{A}$ is assumed to be unital.
$\mathcal{A}^+$: the collection of positive (semidefinite) elements,
$\mathcal{A}^{++}$: the set of all positive definite elements of $\mathcal{A}$, i.e., the set of all invertible elements of $\mathcal{A}^+$.
We call $\mathcal{A}^+, \mathcal{A}^{++}$ the positive semidefinite cone and the positive definite cone of $\mathcal{A}$, respectively.

Kubo-Ando means.
Important theory of means on the positive semidefinite cone of the full operator algebra $B(H)$ over an (infinite dimensional) complex Hilbert space H.

KA means σ correspond to operator monotone functions $f:]0,\infty[\to]0,\infty[$ normalized by the condition $f(1)=1$ via the formula

$$A\sigma B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}, \quad A, B \in B(H)^{++}.$$

Because of the continuous functional calculus, it makes sense in the positive definite cones of general C^*-algebras, too.

In particular, the KA geometric mean $\#$ is defined on the positive definite cone $\mathcal{A}^{++}$ of an arbitrary C^*-algebra $\mathcal{A}$ by

$$A \# B = A^{1/2} (A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}, \quad A, B \in \mathcal{A}^{++}. \tag{1}$$

It is well-known that in the case where $\mathcal{A}$ is the algebra $\mathbb{M}_n$ of all $n \times n$ complex matrices, the geometric mean is closely connected to a natural Riemannian geometry on the corresponding positive definite cone.
Indeed, for any two positive definite matrices $A, B \in \mathbb{M}_n$, the geometric mean $A \# B$ is just the midpoint on the unique geodesic curve

$$t \mapsto A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}, \quad t \in [0,1]$$

connecting A and B in the mentioned Riemannian structure.
The geodesic distance between A and B (the length of the unique geodesic curve connecting A and B) is

$$\| \log A^{-1/2}BA^{-1/2} \|_2,$$

where $\|.\|_2$ stands for the Frobenius norm.

So, in matrix algebras, there is a geometric background behind the geometric mean. What happens in the setting of abstract C^*-algebras?

In that case a Finsler-type structure can be defined on the positive definite cone $\mathcal{A}^{++}$ in which the above curve is length minimizing with length

$$d_T(A, B) = \| \log A^{-1/2} B A^{-1/2} \|, \quad A, B \in \mathcal{A}^{++},$$

where $\|.\|$ stands for the C^*-norm.

That structure was introduced and studied in a series of papers by Corach, Porta and Recht.

Observe that the quantity $d_T(A, B)$ happens to be equal to the so-called Thompson distance between A and B.

So, we have a certain metric structure on the positive definite cone of a C^*-algebra and a corresponding algebraic structure induced by the operation of the geometric mean.

Which are the corresponding preservers, symmetries?

The metric structure:
We gave the precise description of the surjective Thompson isometries of positive definite cones of C^*-algebras, Hatori & M, 2014.

Algebraic structure:
Also gave the description of the (continuous) geometric mean preserving bijective maps between positive definite cones of von Neumann factors, M, 2017. (The problem for general C^*-algebras is open.)

Namely, we proved the following.

If $\mathcal{A}, \mathcal{B}$ are von Neumann algebras and $\mathcal{A}$ is a factor, then the continuous bijective map $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ satisfies

$$\phi(A\#B) = \phi(A)\#\phi(B), \quad A, B \in \mathcal{A}^{++}$$

if and only if
(a) in the case where $\mathcal{A}$ is infinite, ϕ is of the form

$$\phi(A) = T\theta(A^c)T, \qquad A \in \mathcal{A}^{++}.$$

where $\theta : \mathcal{A} \to \mathcal{B}$ is either an algebra *-isomorphism or an algebra *-antiisomorphism, $c \in \{-1, 1\}$, and $T \in \mathcal{B}^{++}$, or
(b) in the case where $\mathcal{A}$ is finite with unique normalized trace Tr, ϕ is of the form

$$\phi(A) = e^{d \operatorname{Tr}(\log A)} T\theta(A^c)T, \qquad A \in \mathcal{A}^{++}$$

where $\theta : \mathcal{A} \to \mathcal{B}$ is an algebra *-isomorphism or an algebra *-antiisomorphism, $c \in \{-1, 1\}$, d is a real number d with $d \neq -c$ and $T \in \mathcal{B}^{++}$.
(Continuity is meant in the norm topology of C^*-algebras.)

The story around the Wasserstein mean.

Bhatia, Jain and Lim have recently introduced a new mean for positive definite matrices that they called Wasserstein mean.

For the positive definite cone of an arbitrary C^*-algebra $\mathcal{A}$ we define this analogously:

$$A\sigma_w B = \frac{1}{4}\left(A + B + A(A^{-1}\#B) + (A^{-1}\#B)A\right), \quad A, B \in \mathcal{A}^{++}. \tag{2}$$

(Note that it is not a KA mean.) It was observed by Bhatia, Jain and Lim that, just like in the case of the KA geometric mean $\#$, there is a geometrical background behind σ_w.

Recall that there is another important metric on the positive definite cone of $\mathbb{M}_n$ that BJL called the Bures-Wasserstein metric:

$$d_{BW}(A, B) = \left(\operatorname{Tr} A + \operatorname{Tr} B - 2\operatorname{Tr}(A^{1/2}BA^{1/2})^{1/2}\right)^{1/2}$$

for any positive definite matrices A, B.

Bhatia, Jain and Lim observed that there is a Riemannian geometry on the positive definite cone of $\mathbb{M}_n$ whose geodesic distance is exactly the metric above and the geodesic curve connecting the points A, B is

$$t \mapsto (1-t)^2 A + t^2 B + t(1-t)(A(A^{-1}\#B) + (A^{-1}\#B)A), \quad t \in [0,1].$$

We see that the midpoint of this curve is just the Wasserstein mean $A\sigma_w B$ of A and B.

The definition of the Bures-Wasserstein metric has recently been extended by Farenick and Rahaman to the setting of C^*-algebras with a faithful finite trace. In 2018 we determined the precise structure of the corresponding surjective isometries.

To sum up:
We have the descriptions of Thompson isometries and geometric mean preserving maps.

Also, we have the description of Bures-Wasserstein isometries.

The missing part of the picture is the determination of the structure of the isomorphisms with respect to the Wasserstein mean.

The results:

In the first theorem we describe the isomorphisms of positive definite cones with respect to the Wasserstein mean. (Quite surprisingly, the solution strongly uses our earlier result concerning the structure of Thompson isometries.)

If one calculates the Wasserstein mean, one obtains the following not easily manageable expression

$$A\sigma_w B = \frac{1}{4}\left(A + B + A^{1/2}(A^{1/2}BA^{1/2})^{1/2}A^{-1/2} + A^{-1/2}(A^{1/2}BA^{1/2})^{1/2}A^{1/2}\right).$$

Still we have the complete structure of isomorphisms.

Theorem 1

Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras and $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ be a bijective map. Then ϕ preserves the Wasserstein mean, i.e., it satisfies

$$\phi(A \sigma_w B) = \phi(A) \sigma_w \phi(B), \quad A, B \in \mathcal{A}^{++} \tag{3}$$

*if and only if there is a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{B}$ and a central element $C \in \mathcal{B}^{++}$ such that*

$$\phi(A) = CJ(A), \quad A \in \mathcal{A}^{++}.$$

Observe that the Bures-Wasserstein isometries have exactly the same form (but with one more condition: preservation of the trace)! In the case of the KA geometric mean and the Thompson metric the situation is very much different...

A bijective linear map $J : \mathcal{A} \to \mathcal{B}$ is called a Jordan *-isomorphism if it satisfies

$$J(AB + BA) = J(A)J(B) + J(B)J(A), \quad J(A)^* = J(A^*), \quad A, B \in \mathcal{A}.$$

Any such map J preserves the Jordan triple product, i.e., satisfies $J(ABA) = J(A)J(B)J(A)$ for all $A, B \in \mathcal{A}$.
For every $A \in \mathcal{A}$, A is invertible if and only if $J(A)$ is invertible and we also have $J(A)^{-1} = J(A^{-1})$.
In particular, J preserves the spectrum of the elements.
The transformation J is compatible with the continuous functional calculus, i.e., for an arbitrary self-adjoint element $A \in \mathcal{A}$ and continuous real function f on the spectrum of A, we have $J(f(A)) = f(J(A))$.

It is trivial to check that every Jordan *-isomorphism J between C^*-algebras maps the positive definite cone onto the positive definite cone and preserves (all Kubo-Ando means as well as) the Wasserstein mean. In particular, the necessity part of Theorem 1 is apparent.

We consider the following related operation:

$$A \circ B = 4(A \sigma_w B), \quad A, B \in \mathcal{A}^{++}.$$

For commuting $A, B \in \mathcal{A}^{++}$, we have $A \circ B = (A^{1/2} + B^{1/2})^2$. In particular, for a commutative algebra $\mathcal{A}$, $\circ$ is an associative operation.

As for the general, noncommutative case, in the view of the complicated formula of the Wasserstein mean, it might be somewhat surprising that the structure $(\mathcal{A}^{++}, \circ)$ has some nice algebraic properties.

It is commutative and, although not associative in general, but the following is true.

Proposition 2

Let $\mathcal{A}$ be a C^*-algebra. For any $A, B \in \mathcal{A}^{++}$ we have

(i) $A \circ (A \circ B) = (A \circ A) \circ B$ *(left alternative identity);*

(ii) $A \circ (B \circ B) = (A \circ B) \circ B$ *(right alternative identity);*

(iii) $A \circ (B \circ A) = (A \circ B) \circ A$ *(flexible identity).*

$(\mathcal{A}^{++}, \circ)$ *is a semigroup if and only if the algebra $\mathcal{A}$ is commutative.*

Using Theorem 1, we can easily determine all automorphisms of $(\mathcal{A}^{++}, \circ)$.

Theorem 3

Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras and $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ be a bijective map. Then ϕ is an $\circ$-isomorphism, i.e., it satifies

$$\phi(A \circ B) = \phi(A) \circ \phi(B), \quad A, B \in \mathcal{A}^{++} \tag{4}$$

*if and only if there is a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{B}$ and a central element $C \in \mathcal{B}^{++}$ such that*

$$\phi(A) = CJ(A), \quad A \in \mathcal{A}^{++}.$$

Linear functionals play a special role among all linear operators in linear analysis. This gives us the motivation that, besides the study of (bijective) mean preservers between positive definite cones of general C^*-algebras, we consider also the corresponding functionals, i.e., scalar valued mean preservers.

Observe that on the positive definite cone of a matrix algebra we do have nontrivial functionals of the geometric mean, e.g., any powers of the determinant function. (Actually, we have the complete description of all such functionals on the positive definite cone of a von Neumann algebra.)

What is the situation with the other "geodesic mean", the Wasserstein mean? The following result shows that the case is significantly different; the existence of a nonconstant such functional typically implies that the underlying algebra has a commutative direct summand.

Theorem 4

Let $\mathcal{A}$ be a von Neumann algebra without type I_2, I_1 direct summands. Any function $F : \mathcal{A}^{++} \to]0, \infty[$ satisfying $F(A\sigma_w B) = F(A)\sigma_w F(B)$, i.e.,

$$F(A\sigma_w B) = \left(\frac{F(A)^{1/2} + F(B)^{1/2}}{2} \right)^2, \quad A, B \in \mathcal{A}^{++} \tag{5}$$

is necessarily constant.

Comments on the appearing conditions: For a commutative algebra $\mathcal{A}$, it is trivial that for any positive linear functional f on $\mathcal{A}$ and nonnegative real number c, the map $A \mapsto (f(A^{1/2}) + c)^2$ satisfies (5).
That is the reason for excluding algebras with type I_1 direct summands.

What concerns the existence of type I_2 direct summands, the reason for excluding that case is that in the proof we apply a former result whose proof depends on the solution of the famous Mackey-Gleason problem which requires that assumption.

The immediate questions arise that what happens in the case where $\mathcal{A} = \mathbb{M}_2(\mathbb{C})$ as well as what happens if $\mathcal{A}$ is a general C^*-algebra.

The basic strategy of the proof of our first theorem is as follows.

We first deduce that the bijective map $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ which satisfies

$$\phi(A \sigma_w B) = \phi(A) \sigma_w \phi(B), \quad A, B \in \mathcal{A}^{++}$$

is necessarily a positive homogeneous order isomorphism between positive definite cones.

Such maps are automatically Thompson isometries! We apply a consequence of our former theorem describing such isometries and obtain that ϕ is necessarily the composition of a Jordan *-isomorphism J and a congruence transformation, i.e., a two sided multiplication by a fixed positive definite element T:

$$\phi(A) = TJ(A)T^*, \quad A \in \mathcal{A}^{++}.$$

We prove that that fixed element T is central.

Key observations to carry out this plan:

I. A characterization of the order (the usual Löwner order) expressed by the operation $\circ$ or equivalently by σ_w.

For any $A, B \in \mathcal{A}_s$ we write $A \leq B$ if $B - A \in \mathcal{A}^+$.

Lemma 5

Let $\mathcal{A}$ be a C^*-algebra. For any $A \in \mathcal{A}^{++}$, set $\mathcal{I}_A = \{A \circ X \ : \ X \in \mathcal{A}^{++}\}$. Given $A, B \in \mathcal{A}^{++}$, we have $\mathcal{I}_A \subset \mathcal{I}_B$ if and only if $B \leq A$.

II. A characterization of the centrality of positive definite elements in C^*-algebras expressed by the commutative operation $\circ$.

Lemma 6

Let $\mathcal{A}$ be a C^*-algebra and $A \in \mathcal{A}^{++}$. We have that A is a lower bound of $\mathcal{I}_A$ if and only if A is a central element of the algebra $\mathcal{A}$.

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Thank you very much for your kind attention!