Closed $G_2$-structures on compact locally homogeneous spaces

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A $G_2$-structure on a 7-manifold $M$ is given by a 3-form $\varphi$ with pointwise stabilizer isomorphic to $G_2$.

- Pointwise $\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$.
- $\varphi$ is non-degenerate: $i_X \varphi \wedge i_X \varphi \wedge \varphi \neq 0$, for every $X \neq 0$.
- $\varphi$ induces a Riemannian metric $g_\varphi$ with volume form $dV_\varphi$:

$$g_\varphi(X, Y) dV_\varphi = \frac{1}{6} i_X \varphi \wedge i_Y \varphi \wedge \varphi.$$
Proposition (Fernández-Gray)

The following are equivalent:

(a) $\nabla^{LC} \varphi = 0$;
(b) $d\varphi = 0$ and $d(\ast \varphi) = 0$;
(c) $Hol(g_\varphi)$ is isomorphic to a subgroup of $G_2$.

A $G_2$-structure satisfying (a), (b) or (c) is called parallel.

Remark

• The conditions $\nabla^{LC} \varphi = 0$ and $d(\ast \varphi) = 0$ are non-linear in $\varphi$.
• Metrics induced by parallel $G_2$-structures are Ricci-flat [Bonan].
A $G_2$-structure $\varphi$ is closed (or calibrated) if $d\varphi = 0$. Then

$$d * \varphi = \tau \wedge \varphi,$$

where $\tau \in \Lambda^2_{14} \cong g_2$, i.e. $\tau \wedge \varphi = - * \tau$ and $\tau \wedge * \varphi = 0$.

**Remark**

- $\tau = d^* \varphi \Rightarrow d^* \tau = 0 \Rightarrow d\tau = \Delta \varphi \varphi$, where $\Delta \varphi = dd^* + d^* d$ is the Hodge Laplacian.

- $\varphi$ defines a calibration on $M$ (i.e. $\varphi|_\xi \leq vol_\xi$, $\forall$ tg oriented 3-plane $\xi$) [Harvey-Lawson].

- $\text{Scal}(g_\varphi) = - \frac{1}{2} |\tau|^2 \leq 0$ [Bryant] $\Rightarrow$ no restrictions on compact manifolds!
General results on the existence of closed $G_2$-structures on (compact) 7-manifolds are still not known.

$$\text{Aut}(M, \varphi) := \{ f \in \text{Diff}(M) | f^* \varphi = \varphi \} \Rightarrow \text{when } M \text{ is compact its Lie algebra is } \text{aut}(M, \varphi) = \{ X \in \chi(M) | L_X \varphi = 0 \}.$$  

**Theorem (Podestá-Raffero)**  

$M$ compact with $\varphi$ closed non-parallel. Then

- $\dim \text{aut}(M, \varphi) \leq b_2(M)$;
- $\text{aut}(M, \varphi)$ is abelian with $\dim \leq 6$.

$\hookrightarrow$ There are no compact homogeneous examples with invariant (non-parallel) closed $G_2$-structures.
The first known example of compact manifold admitting a closed $G_2$-structure but with no parallel $G_2$-structures is a nilmanifold $\Gamma \backslash N$ [Fernández].

**Problem**

*Study the existence of invariant closed $G_2$-structures on compact locally homogeneous $\Gamma \backslash G$, with $G$ Lie group.*

$\Gamma \backslash G$ with an invariant closed $G_2$-structure $\varphi \iff (g, \varphi)$

**Remark**

$g$ has to be *unimodular*, i.e. $\text{tr}(\text{ad}_X) = 0$, for every $X \in g$. 
Classification results on Lie algebras

- **Unimodular non-solvable** Lie algebras [F-Raffero]
  \[ \mathfrak{g} \] must have Levi decomposition \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{r} \) with \( \mathfrak{r} \) centerless if \( \mathfrak{g} \) is a product (3 classes of isomorphism)
  \( \mathfrak{r} \cong \mathbb{R} \ltimes \mathbb{R}^3 \) if \( \mathfrak{g} \) is not a product (1 class of isomorphism).
  \[ \mathfrak{g} \] A unimodular Lie algebra with non-trivial center admitting a closed \( G_2 \)-structure must be solvable!

**Problem**

*Classify all unimodular Lie algebras with non-trivial center admitting closed \( G_2 \)-structures, up to isomorphism.*

- In the nilpotent case there are 12 classes of isomorphisms [Conti-Fernández].
Theorem (F-Raffero-Salvatore)

- There exist 11 isomorphism classes of unimodular non-nilpotent Lie algebras with non-trivial center admitting a closed $G_2$-structure.
- Two of the isomorphism classes are the contactization of a symplectic Lie algebra.

For the proof we use that $\mathfrak{g}$ has to be the central extension of a symplectic Lie algebra $\mathfrak{h}$ endowed with a closed (possibly non-degenerate) 2-form $\omega_0$, i.e. $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}z$, with

$$[z, \mathfrak{h}] = 0, \ [x, y] = -\omega_0(x, y)z + [x, y]_{\mathfrak{h}}, \ \forall x, y \in \mathfrak{h}.$$
Laplacian flow

Idea: use a geometric flow to deform a closed $G_2$-structure and eventually obtain a parallel one

**Definition (Bryant)**

Let $\varphi_0$ be a closed $G_2$-structure on $M^7$. The Laplacian flow (LF) is

$$\begin{cases}
\partial_t \varphi(t) = \Delta \varphi(t) \varphi(t), \\
d \varphi(t) = 0, \\
\varphi(0) = \varphi_0.
\end{cases}$$

where $\Delta \varphi(t)$ is the Hodge Laplacian of $g \varphi(t)$.

If $\varphi(t)$ solves the LF, then $\varphi(t) \in [\varphi_0] \in H^3_{DR}(M^7)$ and

$$\partial_t g_{\varphi(t)} = -2 \text{Ric}(g_{\varphi(t)}) + \text{l.o.t.}$$
Remark

If $M^7$ is compact, then

- stationary points are parallel $G_2$-structures.
- the LF is the gradient flow of Hitchin’s volume functional $V : \varphi \in [\varphi_0] \mapsto \int_M \varphi \wedge \ast \varphi$.

$V$ is monotonically increasing along the LF, its critical points are parallel $G_2$-structures and they are strict local maxima.

Theorem (Bryant-Xu)

Assume that $(M^7, \varphi_0)$ is compact. Then the LF has a unique solution for short time $t \in [0, \epsilon)$, with $\epsilon$ depending on $\varphi_0 = \varphi(0)$. 

Recent developments

• If $\varphi_0$ is near a torsion-free $G_2$-structure $\tilde{\varphi}$, then the LF converges to a torsion-free $G_2$-structure which is related to $\tilde{\varphi}$ via a diffeomorphism [Xu-Ye; Lotay-Wei].

• **Shi-type derivative estimates** for $Rm$ and $\tau$ along the flow:

  a bound on $\Lambda(x, t) := \left( \frac{1}{4} |\nabla \tau|_{g_{\varphi(t)}}^2 + |Rm(x, t)|_{g_{\varphi(t)}}^2 \right)^{1/2}$ will imply bounds on all covariant derivatives of $Rm$ and $\tau$. Then, the flow will exist as long as $\Lambda(x, t)$ remains bounded.

  $\implies$ uniqueness and **compactness** theory [Lotay-Wei].

• **Non-collapsing** under the assumption of bounded $\text{Scal}$ [G. Chen].
Solutions to the LF Study of explicit solutions on

- simply connected Lie groups with left-invariant closed $G_2$-structure [Fernández-F-Manero; Lauret; F-Raffero].

- $\mathbb{T}^7$ with cohomogeneity one closed $G_2$-structure [Huang-Wang-Yao].

- $M^6 \times S^1$, with a warped closed $G_2$-structure $\varphi = f \, ds \wedge \omega + \rho$, $f \in C^\infty(M^6)$, $f > 0$ and compact base $(M^6, \omega, \rho)$ [F-Raffero].

- $M^7$ with an $S^1$-invariant closed $G_2$-structure [Fowdar].

- $(M^4 \times T^3, \varphi)$, where $\varphi$ is induced by a hypersymplectic structure $(\omega_1, \omega_2, \omega_3)$ on the compact $M^4$ [Fine-Yao].

- Long-time existence and convergence of LF flow in cases related to coassociative fibrations [Lambert-Lotay].
Self-similar solutions $\varphi(t) = \sigma(t)f_t^*\varphi$ of the LF $\iff$ closed $G_2$-structures $\varphi$ satisfying

$$\Delta \varphi = \lambda \varphi + L_X \varphi \quad \text{(Laplacian soliton)}$$

for some $\lambda \in \mathbb{R}$ and vector field $X$.

**Definition**

A Laplacian soliton $\varphi$ is called

- **shrinking** if $\lambda < 0$,
- **steady** if $\lambda = 0$,
- **expanding** if $\lambda > 0$. 
Theorem (Lin; Lotay-Wei)

On a compact manifold any Laplacian soliton $\varphi$ (which is not torsion-free) must have $\lambda > 0$ and $X \neq 0$.

In particular, on a compact 7-manifold the only steady Laplacian solitons are given by parallel $G_2$-structures.

Open Problem

$\exists$ expanding Laplacian solitons on compact manifolds?

In the non-compact case:

• $\exists$ steady, shrinking and expanding (homogeneous) solitons [Lauret-Nicolini; F-Raffero; Ball].
• $\exists$ inhomogeneous complete steady and shrinking gradient solitons [Ball; Fowdar].
Any homogeneous Laplacian soliton $\varphi$ on a Lie group $G$ is a semi-algebraic soliton, i.e. $X$ is defined by a 1-parameter group of automorphisms induced by a derivation $D$ of $\mathfrak{g}$ [Lauret].

**Theorem (F, Raffero, Salvatore)**

Let $\mathfrak{g}$ a unimodular Lie algebra with $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$ admitting a semi-algebraic soliton $\varphi$. Then

- if $\mathfrak{g}$ is the contactization of a symplectic Lie algebra, then $\lambda = |\tau|^2 \hookrightarrow \varphi$ is expanding;
- if $\dim \mathfrak{z}(\mathfrak{g}) = 2 \hookrightarrow 10$ isomorphism classes (7 are nilpotent).

**Remark**

The known examples of Lie groups admitting shrinking or steady Laplacian solitons have trivial center!
Remark

An expanding Laplacian soliton is an exact $G_2$-structure!

Problem

*Does there exist compact $\Gamma \backslash G$ with an invariant exact $G_2$-structure?*

A unimodular Lie group cannot admit any left-invariant exact symplectic form [Diatta-Manga].

Example (Fernández-F-Raffero)

There exists a unimodular solvable Lie algebra $\mathfrak{s} = \mathbb{R} \ltimes \mathfrak{n}$, with $\mathfrak{n}$ 4-step nilpotent, satisfying $b_2(\mathfrak{s}) = b_3(\mathfrak{s}) = 0$ and admitting an exact $G_2$-structure.
Remark

The simply connected solvable Lie group $S$ with Lie algebra $\mathfrak{s}$ is not strongly unimodular $\Rightarrow$ $S$ does not admit any compact quotient $\Gamma \backslash S$.

Definition (Garland)

A solvable $G$ is strongly unimodular if $\text{tr}(\text{ad}_X)|_{n^i/n^{i+1}} = 0$, for every $X \in \mathfrak{g}$, where $n^0 = n$, $n^i = [n, n^{i-1}]$, $i \geq 1$, is the descending central series of the nilradical $n$ of $\mathfrak{g}$.

There are no compact examples $\Gamma \backslash G$ with an invariant exact $G_2$-structure, if $\mathfrak{g}$ is (2,3)-trivial, i.e. if $b_2(\mathfrak{g}) = b_3(\mathfrak{g}) = 0$.

Theorem (Fernández-F-Raffero)

A strongly unimodular $(2,3)$-trivial $\mathfrak{g}$ does not admit any exact $G_2$-structure.
To prove the result:

- we use the property that a \((2,3)\)-trivial \(g\) is solvable and 
  \(g = \mathbb{R} \ltimes n\), with \(n\) nilpotent [Madsen-Swann]
- we classify 7-dim strongly unimodular \((2,3)\)-trivial Lie algebras.

**Problem**

*What happens if either \(b_3(g) \neq 0\) or \(b_2(g) \neq 0\)?*

**Theorem (Freibert, Salamon)**

*If the Lie algebra of \(G\) has a codimension-one nilpotent ideal, then \(\Gamma \backslash G\) does not admit any invariant exact \(G_2\)-structure. If in addition \(G\) is completely solvable, \(\Gamma \backslash G\) does not have any exact \(G_2\)-structure at all.*
If $M$ is compact with a closed $G_2$-structure $\varphi$, then

1) $g_\varphi$ Einstein $\Rightarrow \tau \equiv 0$, i.e. $\varphi$ is parallel.

2) $\int_M [\text{Scal}(g_\varphi)]^2 dV_\varphi \leq 3 \int_M |\text{Ric}(g_\varphi)|^2 dV_\varphi$.

[Bryant]: equality in 2) holds if and only if

$$d\tau = \frac{|\tau|^2}{6} \varphi + \frac{1}{6} \ast (\tau \wedge \tau),$$

in such a case, $\varphi$ is called extremally Ricci pinched (ERP).
Theorem (F-Raffero)

*M compact with an ERP closed $G_2$-structure $\varphi$. Then the solution of the Laplacian flow with initial condition $\varphi(0) = \varphi$ is defined for every $t \in \mathbb{R}$ and remains ERP.*

Example (Kath-Lauret)

A compact locally homogeneous space with an ERP $G_2$-structure is given by the compact quotient of the unimodular solvable Lie group $S$ with structure equations

$$(0, 0, 0, -e^{14} - e^{24} - e^{34}, -e^{15} + e^{25} + e^{35}, e^{16} - e^{26} + e^{36}, e^{17} + e^{27} - e^{37})$$

by a lattice.

$S$ is the only unimodular Lie group admitting a left-invariant ERP $G_2$-structure [F-Raffero].
THANK YOU VERY MUCH FOR THE ATTENTION!!