Trivalent dihedrants and bi-dihedrants

Mimi Zhang

School of Mathematical Sciences, Hebei Normal University

Joint work with Professor Jin-Xin Zhou

June 23, 2021

Contents

1. Introduction
2. Trivalent dihedrants
3. Trivalent bi-dihedrants

Definitions

- All graphs considered are finite, connected, simple and undirected.
- A graph is vertex-transitive (edge-transitive, arc-transitive) if its automorphism group acts transitively on its vertices (edges, arcs).
- Cayley graphs: Given a finite group G and an inverse closed subset $S \subseteq G \setminus \{1\}$, the Cayley graph $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$.
- $R(G) = \{R(g) \mid g \in G\} \leq \mathrm{Aut}\,(\mathrm{Cay}(G, S))$.
- $\mathrm{Cay}(G, S)$ is normal if $R(G)$ is normal in $\mathrm{Aut}\,(\mathrm{Cay}(G, S))$.
- A graph is isomorphic to a Cayley graph over $G \iff$ it admits a group isomorphic G as a regular group of automorphisms.

Definitions

- All graphs considered are finite, connected, simple and undirected.
- A graph is vertex-transitive (edge-transitive, arc-transitive) if its automorphism group acts transitively on its vertices (edges, arcs).
- Cayley graphs: Given a finite group G and an inverse closed subset $S \subseteq G \setminus \{1\}$, the Cayley graph $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$.
- $R(G) = \{R(g) \mid g \in G\} \leq \mathrm{Aut}\,(\mathrm{Cay}(G, S))$.
- $\mathrm{Cay}(G, S)$ is normal if $R(G)$ is normal in $\mathrm{Aut}\,(\mathrm{Cay}(G, S))$.
- A graph is isomorphic to a Cayley graph over $G \iff$ it admits a group isomorphic G as a regular group of automorphisms.

Definitions

- All graphs considered are finite, connected, simple and undirected.
- A graph is vertex-transitive (edge-transitive, arc-transitive) if its automorphism group acts transitively on its vertices (edges, arcs).
- Cayley graphs: Given a finite group G and an inverse closed subset $S \subseteq G \setminus \{1\}$, the Cayley graph $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$.
- $R(G) = \{R(g) \mid g \in G\} \leq \mathrm{Aut}\,(\mathrm{Cay}(G, S))$.
- $\mathrm{Cay}(G, S)$ is normal if $R(G)$ is normal in $\mathrm{Aut}\,(\mathrm{Cay}(G, S))$.
- A graph is isomorphic to a Cayley graph over $G \iff$ it admits a group isomorphic G as a regular group of automorphisms.

Definitions

- All graphs considered are finite, connected, simple and undirected.
- A graph is vertex-transitive (edge-transitive, arc-transitive) if its automorphism group acts transitively on its vertices (edges, arcs).
- Cayley graphs: Given a finite group G and an inverse closed subset $S \subseteq G \setminus \{1\}$, the Cayley graph $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$.
- $R(G) = \{R(g) \mid g \in G\} \leq \mathrm{Aut}\,(\mathrm{Cay}(G, S))$.
- $\mathrm{Cay}(G, S)$ is normal if $R(G)$ is normal in $\mathrm{Aut}\,(\mathrm{Cay}(G, S))$.
- A graph is isomorphic to a Cayley graph over $G \iff$ it admits a group isomorphic G as a regular group of automorphisms.

Definitions

- A graph is called a bi-Cayley graph over H if it admits a group isomorphic H as a semiregular group of automorphisms with two vertex-orbits.
- Given a finite group H. Let $R,\ L,\ S \subseteq H$ such that $R^{-1} = R$, $L^{-1} = L$ and $1_H \notin R \cup L$.

  The bi-Cayley graph over H, denoted by $\Gamma = \mathrm{BiCay}(H,\ R,\ L,\ S)$:

  Vertex set: $V(\Gamma) = H_0 \cup H_1$, where $H_i = \{h_i \mid h \in H\}$, $i = 0, 1$.

  Edge set: $E(\Gamma) = E_0 \cup E_1 \cup E_{01}$, where

$$E_0 = \{\{h_0,\ g_0\} \mid gh^{-1} \in R\},$$
$$E_1 = \{\{h_1,\ g_1\} \mid gh^{-1} \in L\},$$
$$E_{01} = \{\{h_0,\ g_1\} \mid gh^{-1} \in S\}.$$

Definitions

- If $|R| = |L| = s$, then $\mathrm{BiCay}(H,\ R,\ L,\ S)$ is said to be an *s-type* bi-Cayley graph.
- A bi-Cayley graph over a cyclic group is simply called a *bicirculant*.
- A bi-Cayley graph over a abelian group is simply called a *bi-abeliant*.
- A Cayley (resp. bi-Cayley) graph on a dihedral group is called a *dihedrant* (resp. *bi-dihedrant*).

Examples

- The smallest vertex-transitive non-Cayley graph:

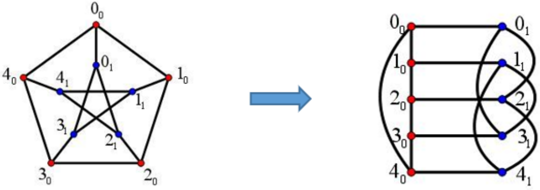


Figure: Petersen graph

$\mathrm{BiCay}(\mathbb{Z}_5, \{1,4\}, \{2,3\}, \{0\})$

Examples

- The smallest vertex-transitive non-Cayley graph:

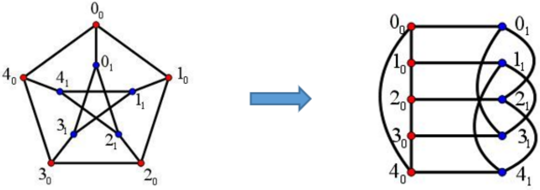


Figure: Petersen graph

$$\mathrm{BiCay}(\mathbb{Z}_5, \{1,4\}, \{2,3\}, \{0\})$$

Examples

- The generalized Petersen graph $P(n,\ t)$

Definition

Let $n \geq 3$ and $1 \leq t \leq n/2$. The generalized Petersen graph $P(n,t)$ is the graph with vertex set $\{\{x_i,\ y_i\} \mid i \in \mathbb{Z}_n\}$ and edge set the union the out edges $\{\{x_i,\ x_{i+1}\} \mid i \in \mathbb{Z}_n\}$, the inner edges $\{\{y_i,\ y_{i+t}\} \mid i \in \mathbb{Z}_n\}$ and the spokes $\{\{x_i,\ y_i\} \mid i \in \mathbb{Z}_n\}$.

- $P(n,\ t) \cong \mathrm{BiCay}(\mathbb{Z}_n, \{1, -1\}, \{t, -t\}, \{0\})$

Examples

- The generalized Petersen graph $P(n,\ t)$

Definition

Let $n \geq 3$ and $1 \leq t \leq n/2$. The generalized Petersen graph $P(n, t)$ is the graph with vertex set $\{\{x_i,\ y_i\} \mid i \in \mathbb{Z}_n\}$ and edge set the union the out edges $\{\{x_i,\ x_{i+1}\} \mid i \in \mathbb{Z}_n\}$, the inner edges $\{\{y_i,\ y_{i+t}\} \mid i \in \mathbb{Z}_n\}$ and the spokes $\{\{x_i,\ y_i\} \mid i \in \mathbb{Z}_n\}$.

- $P(n,\ t) \cong \mathrm{BiCay}(\mathbb{Z}_n, \{1, -1\}, \{t, -t\}, \{0\})$

Examples

Theorem

$P(n,t)$ is vertex-transitive if and only if $t^2 \equiv \pm 1 \pmod{n}$ or $(n,t) = (10,2)$. Moreover, if $t^2 \equiv -1 \pmod{n}$, then $P(n,t)$ is vertex-transitive non-Cayley.

- R. Frucht, J.E. Graver, M.E. Watkins, The groups of the generalized Petersen graphs, Proc. Cambridge Philos. Soc. 70 (1974) 211–218.
- R. Nedela, M. Škoviera, Which generalized Petersen graphs are not Cayley graphs? J. Graph Theory 19 (1995) 1–11.

Background

- D. Marušič and T. Pisanski classified all trivalent vertex-transitive bicirculants;[1] [2]
- J.-X. Zhou and Y.-Q. Feng classfied all trivalent vertex-transitive bi-abeliants.[3]

1 D. Marušič, T. Pisanski, Symmetries of hexagonal molecular graphs on the torus, Croat. Chem. Acta 73 (2000) 969–981.

2 T. Pisanski, A classification of cubic bicirculants, Discrete Math. 307 (2007) 567–578.

3 J.-X. Zhou, Y.-Q. Feng, Cubic bi-Cayley graphs over abelian groups, European J. Combin. 36 (2014) 679–693.

Motivation

Classify trivalent vertex-transitive non-Cayley bi-dihdrants.

By checking the census of trivalent vertex-transitive graphs of order up to 1000,[4] there are 981 non-Cayley graphs, and among these graphs, 233 graphs are non-Cayley bi-dihedrants.

[4] P. Potočnik, P. Spiga, G. Verret, Cubic vertex-transitive graphs on up to 1280 vertices, J. Symb. Comput. 50 (2013) 465–477.

Dihedrants

- In 2000, Marušič and Pisanski gave a classification of trivalent arc-transitive dihedrants.[5]
- S. Du, A. Malnič and D. Marušič gave the complete classification of 2-arc-transitive dihedrants.[6] [7]
- For each prime p, every non-arc-transitive trivalent dihedrant of order $4p$ or $8p$ is either a normal Cayley graph, or isomorphic to the cross ladder graph.[8] [9]

---

[5] D. Marušič, T. Pisanski, Symmetries of hexagonal molecular graphs on the torus, Croat. Chem. Acta 73 (2000) 969–981.

[6] S. Du, A. Malnič, D. Marušič, Classification of 2-arc-transitive dihedrants, J. Combin. Theory B 98 (2008) 1349–1372.

[7] D. Marušič, On 2-arc-transitivity of Cayley graphs, J. Combin. Theory B 87 (2003) 162–196.

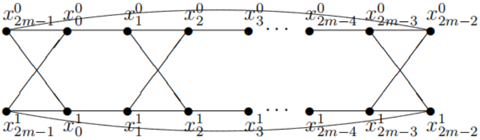
[8] C. Zhou, Y.-Q. Feng, Automorphism groups of connected cubic Cayley graphs of order $4p$, Algebra Colloq. 14 (2007) 351 - 359.

[9] J.-X. Zhou, M. Ghasemi, Automorhisms of a family of cubic graphs, Algebra Colloq. 20 (2013) 495 - 506.

The cross ladder graph

Definition

For an integer $m \geq 2$, the cross ladder graph CL_{4m} has vertex set $V_0 \cup V_1 \cup \ldots V_{2m-2} \cup V_{2m-1}$, where $V_i = \{x_i^0, x_i^1\}$, and edge set $\{\{x_{2i}^r, x_{2i+1}^r\}, \{x_{2i+1}^r, x_{2i+2}^s\} \mid i \in \mathbb{Z}_m, r, s \in \mathbb{Z}_2\}$.



Trivalent dihedrants

Theorem 1 (Zhang &Zhou, AMC, 2021)

Let $\Sigma = \mathrm{Cay}(H, S)$ be a connected trivalent Cayley graph, where $H = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle (n \geq 3)$. If Σ is non-arc-transitive and non-normal, then n is even and $\Sigma \cong \mathrm{CL}_{4 \cdot \frac{n}{2}}$ and $S^\alpha = \{b, ba, ba^{\frac{n}{2}}\}$ for some $\alpha \in \mathrm{Aut}\,(H)$.

Definition

The multi-cross ladder graph, denoted by $\mathrm{MCL}_{4m,2}$, is the graph obtained from CL_{4m} by blowing up each vertex x_i^r of CL_{4m} into two vertices $x_i^{r,0}$ and $x_i^{r,1}$.
The edge set is $\{\{x_{2i}^{r,s}, x_{2i+1}^{r,t}\}, \{x_{2i+1}^{r,s}, x_{2i+2}^{s,r}\} \mid i \in \mathbb{Z}_m, r, s, t \in \mathbb{Z}_2\}$.

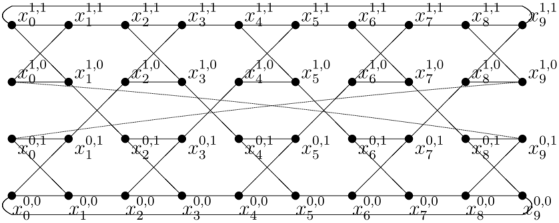


Figure: The multi-cross ladder graph $\mathrm{MCL}_{20,2}$

- Each $\mathrm{MCL}_{4m,2}$ is a bi-Cayley graph.

  $\mathrm{MCL}_{4m,2} \cong \mathrm{BiCay}(H, \{c, ca\}, \{ca, ca^2b\}, \{1\})$, where

  $$H = \langle a, b, c \mid a^m = b^2 = c^2 = 1, a^b = a, a^c = a^{-1}, b^c = b \rangle.$$

- If m is odd, then each $\mathrm{MCL}_{4m,2}$ is a bi-diheddrant. (Let $e = ab$ and $f = ca$)

  $\mathrm{MCL}_{4m,2} \cong \mathrm{BiCay}(H, \{f, fe^{m-1}\}, \{f, fe\}, \{1\})$, where

  $$H = \langle e, f \mid e^{2m} = f^2 = 1, e^f = e^{-1} \rangle.$$

Trivalent bi-dihedrants

- Each $\mathrm{MCL}_{4m,2}$ is a bi-Cayley graph.

  $\mathrm{MCL}_{4m,2} \cong \mathrm{BiCay}(H, \{c, ca\}, \{ca, ca^2b\}, \{1\})$, where

  $$H = \langle a, b, c \mid a^m = b^2 = c^2 = 1, a^b = a, a^c = a^{-1}, b^c = b \rangle.$$

- If m is odd, then each $\mathrm{MCL}_{4m,2}$ is a bi-diheddrant.
  (Let $e = ab$ and $f = ca$)

  $\mathrm{MCL}_{4m,2} \cong \mathrm{BiCay}(H, \{f, fe^{m-1}\}, \{f, fe\}, \{1\})$, where

  $$H = \langle e, f \mid e^{2m} = f^2 = 1, e^f = e^{-1} \rangle.$$

Trivalent bi-dihedrants

- E. Dobson et al. shown that every $\mathrm{MCL}_{4m,2}$ is vertex-transitive.[10]
- J.-X. Zhou and Y.-Q. Feng proved that $\mathrm{MCL}_{4p,2}$ is a vertex-transitive non-Cayley graph for each prime $p > 7$.[11]

Theorem 2 (Zhang &Zhou, AMC, 2021)

The multi-cross ladder graph $\mathrm{MCL}_{4m,2}$ is a Cayley graph if and only if either m is even, or m is odd and $3 \mid m$.

[10] E. Dobson, A. Malnič, D. Marušič, L.A. Nowitz, Semiregular automorphisms of vertex-transitive graphs of certain valencies, J. Combin. Theory B 97 (2007) 371–380.

[11] J.-X. Zhou, Y.-Q. Feng, Cubic vertex-transitive non-Cayley graphs of order $8p$, Electron. J. Comb. 19 (2012) #P53.

Trivalent bi-dihedrants

- E. Dobson et al. shown that every $\mathrm{MCL}_{4m,2}$ is vertex-transitive.[10]
- J.-X. Zhou and Y.-Q. Feng proved that $\mathrm{MCL}_{4p,2}$ is a vertex-transitive non-Cayley graph for each prime $p > 7$.[11]

Theorem 2 (Zhang &Zhou, AMC, 2021)

The multi-cross ladder graph $\mathrm{MCL}_{4m,2}$ is a Cayley graph if and only if either m is even, or m is odd and $3 \mid m$.

[10] E. Dobson, A. Malnič, D. Marušič, L.A. Nowitz, Semiregular automorphisms of vertex-transitive graphs of certain valencies, J. Combin. Theory B 97 (2007) 371–380.

[11] J.-X. Zhou, Y.-Q. Feng, Cubic vertex-transitive non-Cayley graphs of order $8p$, Electron. J. Comb. 19 (2012) #P53.

Trivalent bi-dihedrants

Theorem 3 (Zhang &Zhou, DM, 2017)

Let $H = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle (n \geq 3)$. A connected trivalent bi-dihedrant $\Gamma = \mathrm{BiCay}(H,\ R,\ L,\ S)$ is edge-transitive if and only if the triple (R, L, S) is equivalent to one of the triples in Table 1. Furthermore, all of the graphs in Table 1 are arc-transitive.

Trivalent bi-dihedrants

No.	n	$(R, L, S) \equiv$	Γ	Conditions	Cayley
1	$2m$	$(\{b\}, \{ba^{2t}\}, \{1, a\})$	$CQ(t, m)$	$2 \le t \le m-3$, $m \mid t^2+t+1$	Yes
2	4	$(\{b, ba\}, \{ba^2, ba^3\}, \{1\})$	F016A		Yes
3	4	$(\{b\}, \{ba\}, \{1, a\})$	F016A		Yes
4	5	$(\{b, ba^3\}, \{ba, ba^2\}, \{1\})$	F020B		No
5	5	$(\{b, ba\}, \{a, a^{-1}\}, \{1\})$	F020A		No
6	6	$(\{b, ba\}, \{ba^3, ba^4\}, \{1\})$	F024A		Yes
7	6	$(\{b\}, \{ba^2\}, \{1, a\})$	F024A		Yes
8	8	$(\{b, ba\}, \{ba^2, ba^5\}, \{1\})$	F032A		Yes
9	10	$(\{b, ba^4\}, \{ba, ba^3\}, \{1\})$	F040A		No
10	10	$(\{b, ba^4\}, \{a, a^{-1}\}, \{1\})$	F040A		No
11	12	$(\{b, ba\}, \{ba^3, ba^{10}\}, \{1\})$	F048A		Yes
12	20	$(\{b, ba^{14}\}, \{ba, ba^3\}, \{1\})$	F080A		No
13	$2m$	$(\{b, ba\}, \{ba^{-2t}, ba^{-2t-1}\}, \{1\})$	$CQ(t, m)$	$2 \le t \le m-3$ $m \mid t^2-t+1$	Yes
14	$2m$	$(\{b, ba\}, \{ba^{-2t}, ba^{-2t+m-1}\}, \{1\})$,	$CQ(t, m)$	$2 \le t \le m-3$ $m \mid 2(t^2-t+1)$, m even, t odd	Yes

Table 1: Trivalent edge-transitive bi-dihedrants

Theorem 4 (Zhang &Zhou, DM, 2017)

Every connected trivalent vertex-transitive 0- or 1-type bi-dihedrant is a Cayley graph.

Trivalent bi-dihedrants

For 2-type:

- $H = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle (n \geq 3)$,
- $\Gamma = \mathrm{BiCay}(H,\ R,\ L,\ \{1\})$: a connected trivalent 2-type vertex-transitive bi-Cayley graph over the group H,
- G: a minimum group of automorphisms of Γ subject to $\mathcal{R}(H) \leq G$ and G is transitive on the vertices but intransitive on the arcs of Γ.

Trivalent bi-dihedrants

Theorem 5 (Zhang &Zhou, DM, 2017)

If H_0 and H_1 are blocks of imprimitivity of G on $V(\Gamma)$, then either Γ is Cayley or one of the following occurs:

(1) $(R, L, S) \equiv (\{b,\ ba^{\ell+1}\}, \{ba,\ ba^{\ell^2+\ell+1}\}, \{1\})$, where $n \geq 5$, $\ell^3 + \ell^2 + \ell + 1 \equiv 0 \pmod{n}$, $\ell^2 \not\equiv 1 \pmod{n}$;

(2) $(R, L, S) \equiv (\{ba^{-\ell},\ ba^{\ell}\}, \{a,\ a^{-1}\}, \{1\})$, where $n = 2k$ and $\ell^2 \equiv -1 \pmod{k}$. Furthermore, Γ is also a bi-Cayley graph over an abelian group $\mathbb{Z}_n \times \mathbb{Z}_2$.

Furthermore, all of the graphs arising from (1)-(2) are vertex-transitive non-Cayley.

Trivalent bi-dihedrants

Theorem 6 (Zhang &Zhou, AMC, 2021)

Suppose that H_0 and H_1 are not blocks of imprimitivity of G on $V(\Gamma)$. Then $\Gamma = \mathrm{BiCay}(H, R, L, S)$ is vertex-transitive non-Cayley if and only if one of the followings occurs:

(1) $(R, L, S) \equiv (\{b, ba\}, \{b, ba^{2m}\}, \{1\})$, where $n = 2(2m+1)$, $m \not\equiv 1 \pmod 3$, and the corresponding graph is isomorphic the multi-cross ladder graph $\mathrm{MCL}_{4m,2}$;

(2) $(R, L, S) \equiv (\{b, ba\}, \{ba^{24\ell}, ba^{12\ell-1}\}, \{1\})$, where $n = 48\ell$ and $\ell \geq 1$.

Theorem 7 (Zhang &Zhou, AMC, 2021)

Let $\Gamma = \mathrm{BiCay}(R, L, S)$ be a trivalent vertex-transitive bi-dihedrant where $H = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$ is a dihedral group. Then either Γ is a Cayley graph or one of the following occurs:

(1) $(R, L, S) \equiv (\{b,\ ba\}, \{a,\ a^{-1}\}, \{1\})$, where $n = 5$.

(2) $(R, L, S) \equiv (\{b,\ ba^{\ell+1}\}, \{ba,\ ba^{\ell^2+\ell+1}\}, \{1\})$, where $n \geq 5$, $\ell^3 + \ell^2 + \ell + 1 \equiv 0 \pmod{n}$, $\ell^2 \not\equiv 1 \pmod{n}$.

(3) $(R, L, S) \equiv (\{ba^{-\ell},\ ba^{\ell}\}, \{a,\ a^{-1}\}, \{1\})$, where $n = 2m$ and $\ell^2 \equiv -1 \pmod{m}$. Furthermore, Γ is also a bi-Cayley graph over an abelian group $\mathbb{Z}_n \times \mathbb{Z}_2$.

(4) $(R, L, S) \equiv (\{b, ba\}, \{b, ba^{2m}\}, \{1\})$, where $n = 2(2m+1)$, $m \not\equiv 1 \pmod{3}$, and the corresponding graph is isomorphic the multi-cross ladder graph $\mathrm{MCL}_{4m,2}$.

(5) $(R, L, S) \equiv (\{b, ba\}, \{ba^{24\ell}, ba^{12\ell-1}\}, \{1\})$, where $n = 48\ell$ and $\ell \geq 1$.

Moreover, all of the graphs arising from (1)-(5) are vertex-transitive non-Cayley.

Thanks!