# Spectral aspects of eventually positive $C_0$-semigroups

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Consider the abstract Cauchy problem

$$\dot{u}(t) = Au(t) \quad (t \geq 0), \qquad u(0) = u_0;$$

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**Example (Finite Dimensions)**

On $E = \mathbb{R}^4$, let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

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Then $e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \geq 0$ for all $t \geq 0$.

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Nevertheless, $e^{tA} \geq 0$ for all $t \geq 2$.[1]

[1] Noutsos and Tsatsomeros (2008)

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On $E = L^2(0,1)$, consider $A : u \mapsto u''$ with

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Then $\exists\, t_0 > 0 : e^{tA} \geq 0$ for all $t \geq t_0$ (but not for small $t$).[1]

[1] Daners and Glück (Jun 2018)

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**Example (Dirichlet-to-Neumann)**

Let $\Omega \subseteq \mathbb{R}^2$ : unit disk and $\Delta_D$ : Dirichlet Laplacian on $L^2(\Omega)$.

For $g \in L^2(\partial\Omega)$ and $\lambda \in \mathbb{R} \setminus \sigma(\Delta_D)$, we solve,

$$\Delta f = \lambda f \quad \text{in } \Omega, \qquad f = g \quad \text{on } \partial\Omega.$$

Let $E = L^2(\partial\Omega)$ and for smooth $f$, define $D_\lambda : g \mapsto \frac{\partial f}{\partial \nu}$.

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For large $\lambda$: $e^{-tD_\lambda} \geq 0$ for all $t \geq 0$.

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For large $\lambda$: $e^{-tD_\lambda} \geq 0$ for all $t \geq 0$.

For certain $\lambda$: $e^{-tD_\lambda} \geq 0$ for all large $t$ but not for small $t$.[1]

[1] Daners (2014)

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Even more examples:

- D. Daners, J. Glück, and J. B. Kennedy (Jan & Sep 2016)
- D. Daners and J. Glück (Jun 2018)
- F. Gregorio and D. Mugnolo (2020)
- R. Denk, M. Kunze, and D. Ploß (2021)

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**Definitions**

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- positive if

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- (uniformly) eventually positive if

$$\exists \; t_0 \geq 0 \; \forall f \geq 0 \; \forall \; t \geq t_0 : e^{tA} f \geq 0.$$

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**Proposition**

Let $(e^{tA})_{t\geq 0}$ be a $C_0$-semigroup on a Banach lattice $E$. Then for each $f \in E$,

$$\mathcal{R}(\lambda, A) f = \int_0^\infty e^{-\lambda s} e^{sA} f \, ds$$

for all $\operatorname{Re} \lambda > \omega_0(A)$.

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**Proposition**

Let $(e^{tA})_{t\geq 0}$ be a positive $C_0$-semigroup on a Banach lattice $E$. Then for each $f \in E$,

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Let $(e^{tA})_{t\geq 0}$ be a positive $C_0$-semigroup on a Banach lattice $E$.
Then $s(A) = \omega_0(A)$ when

- $E$ is a Hilbert space.
- $E = L^p(\Omega, \Sigma, \mu)$ for $\sigma$-finite measure spaces $(\Omega, \Sigma, \mu)$.
- $E = C(K)$ or $E = C_0(\Omega)$.

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[1]Daners, Glück, and Kennedy (Jan 2016)

Let $(e^{tA})_{t\geq 0}$ be a $C_0$-semigroup on a Banach lattice $E$.

**Theorem**[1]

If $(e^{tA})_{t\geq 0}$ is positive and bounded, then $\sigma(A) \cap i\mathbb{R}$ is empty or cyclic, i.e., $i\beta \in \sigma(A) \cap i\mathbb{R} \Rightarrow in\beta \in \sigma(A) \cap i\mathbb{R}$ for all $n \in \mathbb{N}$.

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**Theorem**[2]

Suppose $(e^{tA})_{t\geq 0}$ is eventually positive and for each $f \in E$, the orbit $\{e^{tA}f : t \geq 0\}$ is relatively compact in weak topology.

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Application[3]: Let $E = L^2(0,1), A : u \mapsto u'''$ with periodic boundary conditions.

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Suppose $(e^{tA})_{t\geq 0}$ is positive, $s(A) = 0$, and $(e^{tA})_{t\geq 0}$ is bounded.

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Suppose $(e^{tA})_{t\geq 0}$ is positive, $s(A) = 0$, and $(e^{tA})_{t\geq 0}$ is bounded.

Then $\sigma(A) \cap i\mathbb{R} = \{0\}$.

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Let $(e^{tA})_{t\geq 0}$ be an eventually norm continuous $C_0$-semigroup on a Banach lattice $E$.

**Theorem (Strong convergence)**

If $E$ is reflexive and $(e^{tA})_{t\geq 0}$ is positive and bounded, then $\lim_{t\to\infty} e^{tA}f$ exists for all $f \in E$.

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Proof: Cyclicity + the ABLV theorem[1].

[1] Arendt and Batty (1988), Lyubich and Vũ (1988)

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**Theorem (Strong Convergence)**[1]

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Proof: Relies heavily on a cyclicity result for single operators[2] + the ABLV theorem.

[1] A. and Glück (2021)

[2] Glück (2017)

Let $(e^{tA})_{t\geq 0}$ be an eventually norm continuous $C_0$-semigroup on a Banach lattice $E$.

**Theorem (Operator norm convergence)**[1]

Suppose $(e^{tA})_{t\geq 0}$ is positive, $s(A) = 0$, and

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## A glimpse of local eventual positivity

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Then $e^{tA}$ converges uniformly as $t \to \infty$.

[1] A. (2021)

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