Ternary self-distributive cohomology and invariants of framed links and knotted surfaces with boundary

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This is based on joint work with:

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- Viktor Abramov

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Organization of the talk:

- Recollections from quandles and cocycle invariants;
- Ternary quandles and their cohomology;
- Ternary cocycle invariants of framed links;
- Ternary cocycle invariants of compact surfaces with boundary;
- Ternary quandles in symmetric monoidal categories;
- Examples from Lie algebras and Hopf monoids;
- Quantum invariants.

(Binary) Quandles

Definition

A quandle is a set X togehter with a binary operation $*: X \times X \longrightarrow X$ satisfying the following three axioms

- $x * x = x$, for all $x \in X$,
- the right multiplicaiton map $- * x : X \longrightarrow X$ is a bijection for all $x \in X$, where $-$ is a placeholder,
- $(x * y) * z = (x * z) * (y * z)$, for all $x, y, z \in X$.

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Remark

The three axioms in the definition of quandle correspond to Reidmeister moves of type I, II and III.

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- Any $\Lambda(= \mathbb{Z}[t, t^{-1}])$-module M is a quandle with $a * b := ta + (1 - t)b$, for $a, b \in M$, and is called an *Alexander quandle*.
- Given a group G and an automorpism $f \in \mathrm{Aut}(G)$, it is easy to show that $x * y := f(xy^{-1})y$ defines a quandle structure. This is called a *generalized Alexander quandle*.

Quandles give rise to Yang-Baxter operators, i.e. maps that satisfy the Yang-Baxter equation:

$$(R \otimes \mathbb{1})(\mathbb{1} \otimes R)(R \otimes \mathbb{1}) = (\mathbb{1} \otimes R)(R \otimes \mathbb{1})(\mathbb{1} \otimes R),$$

by the assignment $(x, y) \mapsto (y, x * y)$ and linearization.

(Co)homology (Carter-Kamada-Saito)

- Define $C_n(X)$ to be the free abelian group generated by n-tuples $(x_1, x_2, \cdots, x_n)$ of elements of a rack X.
- Define differentials $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ as:

$$\begin{aligned} &\partial_n(x_1, x_2, \cdots, x_n) \\ &= \sum_{i=2}^{n} (-1)^i [(x_1, \cdots, \hat{x}_i, \cdots, x_n) \\ &\quad -(x_1 * x_i, \cdots, x_{i-1} * x_i, \hat{x}_i, \cdots, x_n)]. \end{aligned}$$

(Co)homology (Carter-Kamada-Saito)

- Put $H_n = ker\partial_n / im\partial_{n+1}$.
- Normalize the chain complex for quandles.
- Dualize to obtain cohomology.

- Define coloring of a diagram by quandle: $\mathcal{C} : R \longrightarrow X$, where R is the set of arcs of the diagram D with compatibility property similar to rule to define Yang-Baxter operator from quandle.

Cocycle invariant of links

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- Fix 2-cocycle $\phi \in Z^2(X, A)$.
- Then the Boltzmann weight at crossing τ is given by $B(\tau, \mathcal{C}) := \phi(x, y)^{\epsilon(\tau)}$.
- Set $\Psi = \sum_{\mathcal{C}} \prod_{\tau} B(\tau, \mathcal{C})$.

Theorem

The Boltzmann state sum defined above is invariant under Reidmeister moves. It therefore defines an invariant of links.

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If we compute the cocycle invariant with respect to two cocycles that are in the same cohomology class, then the invariants coincide (up to an integer term).

Ternary racks/quandles

- A set X together with a ternary operation $T : X \times X \times X \longrightarrow X$ satisfying the properties:
  - $T(T(x, y, z), u, v) = T(T(x, u, v), T(y, u, v), T(z, u, v))$ for all $x, y, z, u, v \in X$.
  - The map $T(-, y, z) : X \longrightarrow X$ is a bijection for all $y, z \in X$.
  - $T(x, x, x) = x$ for all $x \in X$. .

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- Examples:
  - Iteration of binary self distriutive operation: $T(x, y, z) = (x * y) * z$.
  - Heap of a group: $T(x, y, z) = xy^{-1}z$.
  - Mutually distributive operations (Przytycki) composed appropriately.

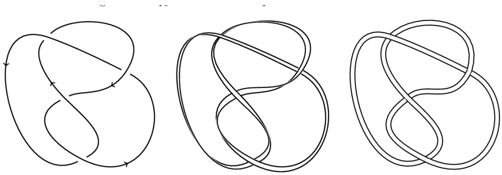
Ternary (co)homology

- Define $C_n(X)$ to be the free abelian group generated by $(2n+1)$-tuples $(x_0, x_1, \cdots, x_{2n})$ of elements of a ternary rack X.
- Define differentials $\partial_n C_n(X) \longrightarrow C_{n-1}(X)$ as:

$$\begin{aligned}
&\partial_n(x_0, x_1, \cdots, x_{2n}) \\
&= \sum_{i=1}^{2n-1} (-1)^i [(x_1, \cdots, \hat{x}_i, \hat{x}_{i+1}, \cdots, x_n) \\
&\quad -(T(x_0, x_i, x_{i+1}), \cdots, T(x_{i-1}, x_i, x_{i+1}), \hat{x}_i, \hat{x}_{i+1}, \cdots, x_n)].
\end{aligned}$$

Framed Knot/Link Invariants

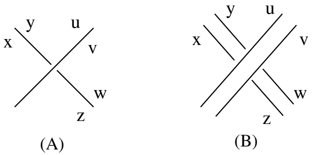
Recall (Framed) Knot Diagrams:



(A) Knot Diagram (B) Framed Knot (C) Blackboard Framing

Figure: Taken from Even-Zohar, Chaim. The writhe of permutations and random framed knots. Random Struct. Algorithms 51 (2017): 121-142.

At crossings:



Generators: all arcs of a diagram. Relators: Defined through equalities $z = T(x, u, v)$ and $w = T(y, u, v)$.

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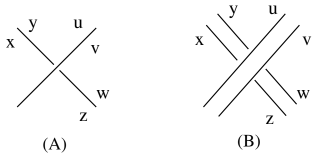

Generators: all arcs of a diagram. Relators: Defined through equalities $z = T(x, u, v)$ and $w = T(y, u, v)$.
Obtain a group whose heap is an invariant of framed links, called *fundamental heap*.

- Define colorings of of framed diagrams: Heap morphisms from fundamental heap.
- Define Boltzmann weights using diagrammatic interpretation at crossings.

Theorem

The Boltzmann sum

$$\Theta(\mathcal{D}) = \sum_{\mathcal{C}} \prod_{\tau} \mathcal{B}(\phi, \tau, \mathcal{C}) \in \mathbb{Z}[A] \otimes \mathbb{Z}[A]$$

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is an invariant of framed links.

Proof.

One needs to show invariance of moves for framed link diagram, instead of Reidemeister moves. RII and RIII are related to functoriality (described above), and kinks with opposite signs give contributions that cancel each other. □

Computations of cohomology show that

- Can define subcomplexes fitting in long exact sequences: Corresponding cohomologies are nontrivial, actually there is abundance of nontrivial cocycles.
- Cohomologies of cyclic heaps, and dihedral heaps, have arbitrarily large ranks.
- The associated ternary cocycle invariants are nontrivial (this can be used to prove lower bounds for rank of cohomology).

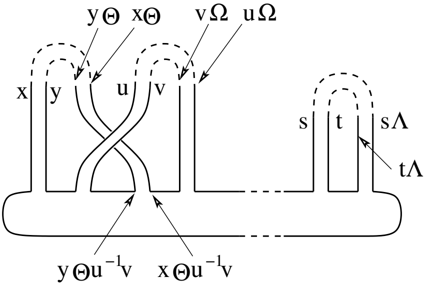

Figure: Normal form of surface ribbons

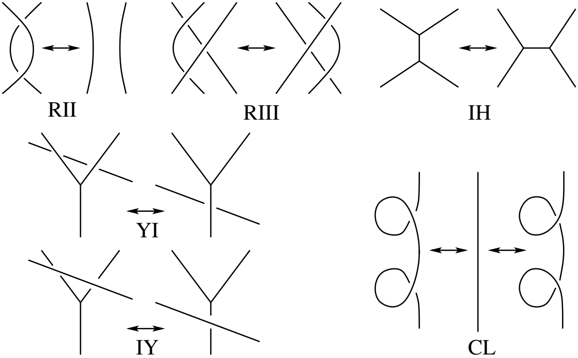

Figure: Moves

- One can define the fundamental heap for surface ribbons (similar procedure as with framed links), and this is an isotopy invariant.

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- Interesting relations between minimal number of generators of fundamental heap, and Euler characteristic and genus of surface ribbons.
- Using stabilization, one can always modify a surface to get a new one whose fundamental heap is free.
- Every heap is realizable as the fundamental heap of some ribbon surface (up to a free factor determined by the number of generators and relators in a given presentation).

- Cohomology with some extra conditions:
  - $\psi(w, x, y) + \psi(wx^{-1}y, y, x) = 0$ (reversibility);
  - $\psi(w, x, y) + \psi(wx^{-1}y, y, z) = \psi(w, x, z)$ (additivity).

Cocycle invariants of ribbon surfaces

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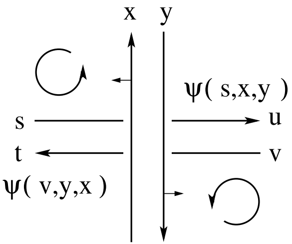
  - $\psi(w, x, y) + \psi(wx^{-1}y, y, z) = \psi(w, x, z)$ (additivity).
- Define Boltzmann sums for each boundary component, where cocycles are used at each crossing:



- Take tensor product of all boundary components and sum over colorings.
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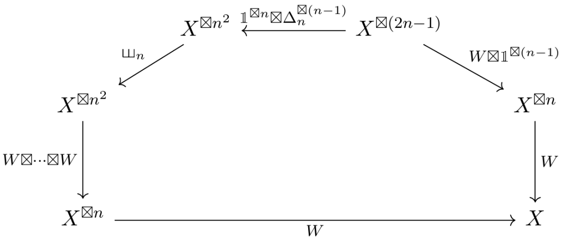
Some results related to cocycle invariants:

- Boundary connected sum can be detected through cocycle invariant (nice formulas relating invaraints of surface ribbons and their connected sums).
- Can prove geometrically that RA cohomology of generalized quaternion groups have free factors.

Categorical self-distributivity

In symmetric monoidal category:



Observe that we need comultiplication Δ (there are repeated elements in the definition of self-distributivity.)

Lie examples

- For a given Lie algebra L over a ground field $\mathbb{k}$, define $X = \mathbb{k} \oplus L$. The map $T : X \otimes X \otimes X \longrightarrow X$, $(a, x) \otimes (b, y) \otimes (c, z) \longmapsto (abc, bcx + c[x, y] + b[x, z] + [[x, y], z])$, extended by linearity, is self distributive.

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- We can also use ternary Lie brackets instead of iterated binary Lie brackets.
- Homotopy Lie algebras!

Hopf Examples

- Let H be a Hopf algebra. Define a ternary operation $T: H \otimes H \otimes H \longrightarrow H$ by the assignment $T(x \otimes y \otimes z) = \mu(\mu(x \otimes S(y)) \otimes z)$, extended by linearity, where μ indicates the product of H and S is the antipode. (Quantum Heap)

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- Let $p : X^{\otimes 2} \longrightarrow H$ be a ternary augmented rack. Then the ternary operation defined on monomials via $x \otimes y \otimes z \longmapsto x \cdot p(y \otimes z)$, and extended by linearity, is self distributive.

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In vector spaces:

$$R(x \otimes y \otimes z \otimes w) = z^{(1)} \otimes w^{(1)} \otimes T(x \otimes z^{(2)} \otimes w^{(2)}) \otimes T(y \otimes z^{(3)} \otimes w^{(3)}).$$

One can construct a category $\mathcal{R}_\alpha(X)$, from a ternary self-distributive object (X, T) in the category of vector spaces, and endow it with a braiding c^α and a nontrivial twist θ^α.

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Theorem

The category $\mathcal{R}^*_\alpha(X)$ with braiding induced by c^α and twisting morphisms induced by θ^α is a ribbon category. Moreover, if $[\alpha] = [\beta]$ the two categories $\mathcal{R}^*_\alpha(X)$ and $\mathcal{R}^*_\beta(X)$ are equivalent.

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The previous category gives rise to an invariant of framed links, $\Psi_{\mathcal{D}}(X, T, \alpha)$, as the quantum trace of an endomorphism of $\mathcal{R}^*_\alpha(X)$, associated to a framed braid representing the framed link.

Theorem

Fix a diagram $\mathcal{D}$ of L. Then the ribbon cocycle invariant $\Theta_{\mathcal{D}}(X, T, \alpha)$ and the quantum invariant $\Psi_{\mathcal{D}}(X, T, \alpha)$ coincide.

The end

Thank you very much for your attention!