Irregular solutions of the transport and Navier-Stokes equations

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We want to describe the motion of some particles of clouds. We model the clouds as a gas/liquid with given velocity \( \mathbf{v}(x) \) for each position \( x \) (direction and intensity).

A single particle is transported along an integral curve of \( \mathbf{v} \)

\[
\frac{d}{dt}\gamma(t) = \mathbf{v}(\gamma(t)) \quad \text{for any } t \in [0, \infty).
\]
If the particles are many, we model them as a distribution, namely with a measure $\mu_0$. $\mu_t$ evolves according to the PDE

$$\partial_t \mu_t + \mathbf{v} \cdot \nabla \mu_t = 0.$$
Incipit - Evolution of a distribution of particles

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Outline of the talk

1. Flow of vector fields and continuity equation

2. Smooth vs nonsmooth theory
   - Lack of uniqueness of the flow for nonsmooth vector fields
   - Regular Lagrangian Flows and the nonsmooth theory

3. A.e. uniqueness of integral curves
   - Ambrosio’s superposition principle
   - Ill-posedness of CE by convex integration
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1. Flow of vector fields and continuity equation
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The flow of a vector field

Given a vector field \( b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \), consider the flow \( X \) of \( b \):

\[
\begin{cases}
\frac{d}{dt} X(t, x) = b_t(X(t, x)) & \forall t \in [0, \infty) \\
X(0, x) = x.
\end{cases}
\]

It can be seen
- as a collection of trajectories \( X(\cdot, x) \) labelled by \( x \in \mathbb{R}^d \);
- as a collection of diffeomorphisms \( X(t, \cdot) : \mathbb{R}^d \to \mathbb{R}^d \).
Consider the related PDE, named **continuity equation**

\[
\begin{cases}
\partial_t \mu_t + \text{div} (b_t \mu_t) = 0 & \text{in } (0, \infty) \times \mathbb{R}^d \\
\mu_0 \text{ given.}
\end{cases}
\]

When $b_t$ is sufficiently smooth and $\mu_t : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ is a smooth function, all derivatives can be computed. Much less is needed to give a distributional sense to the PDE (e.g. $b_t$ bounded and $\mu_t$ finite measures).

When

\[\text{div} b_t \equiv 0,\]

the continuity equation is equivalent to the **transport equation**

\[
\partial_t \mu_t + b \cdot \nabla \mu_t = 0.
\]
Connection between continuity equation and flows

Solutions of the CE flow along integral curves of $b$

Given $b$, its flow $X$ an initial distribution of mass $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$, a solution of the CE is

$$\mu_t := X(t, \cdot) \# \mu_0.$$

Recall that the measure $X(t, \cdot) \# \mu_0$ is defined by

$$\int_{\mathbb{R}^d} \varphi(x) \, d[X(t, \cdot) \# \mu_0](x) = \int_{\mathbb{R}^d} \varphi(X(t, x)) \, d\mu_0(x) \quad \forall \varphi : \mathbb{R}^d \to \mathbb{R}. $$

Indeed, for any test function $\varphi \in C^\infty_c(\mathbb{R}^d)$ we have

$$\frac{d}{dt} \int_{\mathbb{R}^d} \varphi \, d\mu_t = \frac{d}{dt} \int_{\mathbb{R}^d} \varphi(X(t, x)) \, d\mu_0(x) = \int_{\mathbb{R}^d} \nabla \varphi(X) \cdot \partial_t X \, d\mu_0$$

$$\quad = \int_{\mathbb{R}^d} \nabla \varphi(X) \cdot b_t(X) \, d\mu_0 = \int_{\mathbb{R}^d} \nabla \varphi \cdot b_t \, d\mu_t.$$

This is the distributional formulation of the continuity equation.
Is the solution of the continuity equation starting from \( \mu_0 \) unique?

**YES if \( \nabla b \) is bounded**

Given a solution \( \nu_t \) to CE, set \( \tilde{\nu}_t = X(t, \cdot)^{-1} \# \nu_t \) and compute

\[
\frac{d}{dt} \int_{\mathbb{R}^d} \varphi \, d\tilde{\nu}_t = 0,
\]

so \( X(t, \cdot)^{-1} \# \nu_t = \tilde{\nu}_t = \nu_0 = \mu_0 \Rightarrow \nu_t := X(t, \cdot) \# \mu_0. \)

**Cauchy-Lipschitz Theorem**

Let \( b_t \) a vector field with \( \nabla b_t \) bounded. Then for every \( x \in \mathbb{R}^d \) there exists a unique solution \( X(\cdot, x) : [0, \infty) \to \mathbb{R}^d \) of the ODE.

**NO if \( b \) is less regular**

As soon as uniqueness for the ODE fails.
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Less regular vector fields appear

- in **fluid dynamics**, when a fluid or a gas develop a turbulent behavior or a discontinuity/singularity (shear flows, shock waves...). As an example, in the theory of turbulence, the Onsager conjecture regards Holder continuous solutions to Euler; some of the optimal regularity estimates for Navier-Stokes are based on the understanding of its flow.

- in **meteorology**, to build solutions of the semigeostrophic system in 2d and 3d [Ambrosio, C., De Philippis, Figalli, '12, '14];

- in **kinetic equations**, to give a lagrangian description of solutions to the Vlasov-Poisson system [Ambrosio, C., Figalli, '15, '17];

- studying the geometry of nonsmooth manifolds with curvature bounds (in this direction, see also [C., Tione '20]).
Nonsmooth theory: lack of uniqueness

One-dimensional autonomous vector field with lack of uniqueness

\[ b(x) = 2\sqrt{|x|}, \quad x \in \mathbb{R} \]

Given \( x_0 = -c^2 < 0 \), the 1-parameter family of curves that stop at the origin for an arbitrary time \( T \geq 0 \), solve the ODE.

\[ x(t) = \begin{cases} x_0 & \text{if } t < c + T \\ \frac{(t - c - T)^2}{2} & \text{if } t \geq c + T \end{cases} \]

\[ x_0 = -c^2 \]
Between all the possible integral curves, a “better selection” could be made by the ones that do not stop in 0. In other words, we wish to select a collection of integral curves that “do not concentrate”.
Regular lagrangian flows

Given a vector field \( \mathbf{b} : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d \), the map \( \mathbf{X} : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \) is a regular Lagrangian flow of \( \mathbf{b} \) if:

(i) for \( \mathcal{L}^d \)-a.e. \( x \in \mathbb{R}^d \), \( \mathbf{X}(\cdot, x) \) solves the ODE \( \dot{x}(t) = \mathbf{b}_t(x(t)) \) starting from \( x \);

(ii) \( \mathbf{X}(t, \cdot) \# \mathcal{L}^d \leq C \mathcal{L}^d \) for every \( t \in [0, T] \) and for some \( C > 0 \).

Theorem ([Di Perna-Lions '89], [Ambrosio '04])

Let us assume that \( |\nabla \mathbf{b}_t| \in L^1_{loc}(\mathbb{R}^d) \), \( \text{div} \mathbf{b}_t \in L^\infty(\mathbb{R}^d) \) and

\[
\frac{|\mathbf{b}_t(x)|}{1 + |x|} \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d).
\]

Then there exists a unique regular Lagrangian flow \( \mathbf{X} \) of \( \mathbf{b} \).
The regularity assumption $|\nabla b_t| \in L_{loc}^1(\mathbb{R}^d)$ can be replaced by

- $\nabla b_t$ is a matrix-valued finite measure, [Ambrosio 04];
- singular integrals of $L^1$ functions, [Bohun, - Bouschut, Crippa 13].

The assumption $\text{div } b_t \in L^\infty(\mathbb{R}^d)$ can be weakened to $\text{div } b_t \in BMO(\mathbb{R}^d)$ [Mucha, 2010], [C., Crippa, Spirito 2016].

A different approach to this result was proposed by [Crippa, De Lellis, 08], considering a functionals of the type

$$\Phi_\delta(t) := \int \log \left( 1 + \frac{|X_1(t, x) - X_2(t, x)|}{\delta} \right) dx \quad t \in [0, T];$$

**Question: a.e. uniqueness of integral curves**

Does any divergence free $b \in L^1_t W^{1,p}_x$ admit a unique integral curve (namely, $\gamma \in W^{1,1}(0, T)$ solution of the ODE $\dot{\gamma}(t) = b(t, \gamma)$) for a.e. initial datum $x \in \mathbb{R}^d$?

Open since the pioneering works of DiPerna-Lions and Ambrosio.
Flows of vector fields

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Flow of vector fields and continuity equation

Smooth vs nonsmooth theory

Lack of uniqueness

Regular Lagrangian flows

A.e. uniqueness of integral curves

Ambrosio’s superposition principle

Ill-posedness of CE by convex integration
Main result

If $p < d$ then the a.e. uniqueness for trajectories does not hold.

**Theorem ([Brué-C.-DeLellis, ’20])**

For every $d \geq 2$, $p < d$ and $s < \infty$ there exist a divergence free velocity field $b \in C_t(W^{1,p}_x \cap L^s_x)$ and a set $A \subset \mathbb{T}^d$ such that

- $\mathcal{L}^d(A) > 0$;
- for any $x \in A$ there are at least two integral curves of $b$ starting at $x$.

[Sorella, Pitcho, ’21] and [Sorella, Giri, ’21] show that the set $A$ can be taken of full measure in the torus and that the theorem adapts to hamiltonian structures.

What about the critical case $p = d$?
Our strategy

Ingredients of proof:

- Ambrosio’s superposition principle to connect the a.e. uniqueness of trajectories to uniqueness results for positive solutions to (CE).

- A new (asymmetric) Lusin-Lipschitz type inequality.

- Non-uniqueness theorem for positive solutions to (CE) based on convex integration type techniques borrowed from [Modena-Székelyhidi ’18].
Take a vector field $\mathbf{b}$ with two different flows. Then we observed that both

$$X_1(t, \cdot) \# \mu_0 \quad \text{and} \quad X_2(t, \cdot) \# \mu_0$$

solve the CE starting from $\mu_0$. By linearity,

$$\lambda X_1(t, \cdot) \# \mu_0 + (1 - \lambda) X_2(t, \cdot) \# \mu_0$$

is a solution as well. We can interpret this as "choosing $X_1$ with probability $\lambda$ and $X_2$ with probability $1 - \lambda"."
Ambrosio’s superposition principle

A measure valued solution \( \mu \in L^\infty_t(\mathcal{M}_+) \) to (CE) with velocity \( \mathbf{b} \) is a superposition solution if for \( \mu_0 \)-a.e. \( x \in \mathbb{T}^d \) there exists \( \eta_x \in \mathcal{P}(C([0, T], \mathbb{T}^d)) \) such that

- \( \eta_x \) is concentrated on integral curves of \( \mathbf{b} \) starting at \( x \);
- we have the representation formula \( \mu = (e_t)_\#(\mu_0 \otimes \eta_x) \),

\[
\int \phi \, d\mu_t = \int \left( \int \phi(\gamma(t)) \, d\eta_x(\gamma) \right) \, d\mu_0(x).
\]

Superposition solutions are averages of integral curves of \( u \).

Theorem ([Ambrosio ’04])

Let \( \mathbf{b} : [0, T] \times \mathbb{T}^d \to \mathbb{R}^d \), \( \mu \in L^\infty_t(\mathcal{M}_+) \) solution of CE with

\[
\int_0^T \int |\mathbf{b}(t, x)| \, d\mu_t(x) \, dt < \infty.
\]

Then it is a superposition solution.
If we produce an example of nonuniqueness of positive solutions of the continuity equation in some range of exponents we have disproved the a.e. uniqueness of integral curves.

**Theorem ([Brué-C.-DeLellis, ’20] )**

Let \( d \geq 2, \ p \in (1, \infty), \ r \in [1, \infty], \ \frac{1}{r} + \frac{1}{r'} = 1 \) be such that

\[
\frac{1}{p} + \frac{1}{r} > 1 + \frac{1}{d}.
\]

Then there exist \((b, u)\) solution of the CE with
- a divergence-free vector field \( b \in C_t(W^{1,p}_x \cap L^{r'}_x) \),
- a positive, nonconstant \( u \in C_tL^r_x \) with \( u(0, \cdot) = 1 \)
The main theorem follows: any velocity field obtained in the previous theorem does not have the a.e. uniqueness for integral curves. Indeed

- Since $\text{div} \, b = 0$, the function $\bar{u} \equiv 1$ solves CE.
- The $u$ constructed in this theorem is a second distinct solution!
- As seen before, a.e. uniqueness of integral curves implies uniqueness of positive solutions to (CE).

The construction is based on convex integration scheme, as in the groundbreaking works [DeLellis-Székelyhidi, '09-'13], [Isett '16] for the Euler equation and [Buckmaster-Vicol '17] for Navier-Stokes.

The first ill-posedness result for (CE) with Sobolev velocity field has been proven in [Modena-Székelyhidi, '18], [Modena-Sattig, '19].

Main novelties: positive solutions, a simpler convex integration scheme in any dimension.
The convex integration scheme

- We start from CE solved with an error

\[
\begin{cases}
\partial_t u_q + \text{div} (b_q u_q) = \text{div} R_q \\
\text{div} b_q = 0
\end{cases}
\]

Solutions are obtained through an inductive procedure as
\[u = \lim_{q \to \infty} u_q, \quad u = \lim_{q \to \infty} b_q\]
and \(\lim_{q \to \infty} \|R_q\|_{L^1} = 0\).

- We look for \(b_{q+1} = b_q + a B_{q+1}, \quad u_{q+1} = u_q + b U_{q+1}\), where \(B_q\) and \(U_q\) are "highly oscillating" time-dependent versions of Mikado-flows (Cf. [Daneri-Székelyhidi '17]). \(a\) and \(b\) are "slow" functions. They cancel the error when interact

\[|R_q - abB_q U_q| \ll 1.\]

- We exploit the scaling invariances of the equation by making \(B_q\) and \(U_q\) concentrated ([Buckmaster-Vicol, '17]).

Heuristic idea: Ill-posedness happens when \(u\) "concentrates" where \(b\) is far from being Lipschitz (i.e. \(\nabla b\) is "big").
Thank you for your attention! *

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