J-trajectories in Sol_0^4

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8th European Congress of Mathematics

Portorož, June 22, 2021

Contents

1. Preliminaries

Definition 1

A Riemannian manifold (M, g) is said to be **homogeneous** *if for every two points p and q in M, there exists an isometry of M, mapping p into q.*

1982 W.Thurston $\longrightarrow$ "geometrization conjecture"

The eight simply connected 3-dim homogeneous spaces ("model geometries"):

$$E^3,\ S^3,\ H^3,\ S^2 \times \mathbb{R},\ H^2 \times \mathbb{R},\ \mathrm{Nil},\ \mathrm{Sol},\ \widetilde{SL(2,\mathbb{R})}$$

W. M. Thurston, *Three-dimensional Geometry and Topology* I, Princeton Math. Series., vol. **35** (S. Levy ed.), 1997.

Thurston geometries

1-dim Thurston geometry

- $\mathbb{E}^1$

2-dim Thurston geometries

- $\mathbb{E}^2$, $\mathbb{H}^2$, $\mathbb{S}^2$

3-dim Thurston geometries

- the constant sectional curvature geometries: $\mathbb{E}^3$, $\mathbb{H}^3$, $\mathbb{S}^3$
- the product geometries: $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$
- the twisted product geometries: Nil, Sol, $\widetilde{SL(2,\mathbb{R})}$

P. Scott, *The Geometries of 3-Manifolds*, Bull. London Math. Soc., **15** (1983), 401-487.

E. Molnár, *The projective interpretation of the eight 3-dimensional homogeneous geometries*, Beiträge Algebra Geom. **38 (2)** (1997), 261–288.

4-dim Thurston geometries

R.O. Filipkiewicz, *Four dimensional geometries*, PhD Thesis, University of Warwick, 1984.

Nineteen 4-dim Thurston geometries

- $\mathbb{E}^4$, $\mathbb{H}^4$, $\mathbb{S}^4$, $\mathbb{P}^2(\mathbb{C})$, $\mathbb{H}^2(\mathbb{C})$
- $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{S}^2 \times \mathbb{E}^2$, $\mathbb{S}^2 \times \mathbb{H}^2$, $\mathbb{E}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$
- $\mathbb{S}^3 \times \mathbb{E}^1$, $\mathbb{H}^3 \times \mathbb{E}^1$, $\widetilde{SL_2} \times \mathbb{E}^1$, $Nil^3 \times \mathbb{E}^1$
- Nil^4, $Sol^4_{m,n}$, Sol^4_0, Sol^4_1, F^4

C.T.C. Wall, *Geometric structures on compact complex analytic surfaces*, Topology **25**, (1986), 119-152.

- in most cases (14), a geometric structure carries preferred complex structure

Kähler structure

Definition 2

A **Kähler structure** *on a Riemannian manifold* (M, g) *is given by a two-form* Ω *and a field of endomorphisms of the tangent bundle* J *satisfying the following conditions:*

- J *is an almost complex structure:* $J^2 = -I$
- *metric* g *is compatible with* J*:* $g(X, Y) = g(JX, JY), \quad \forall X, Y \in TM$
- *the fundamental (Kähler) form* $\Omega(X, Y) := g(JX, Y)$

- *the 2-form* Ω *is symplectic:* $d\Omega = 0$
- J *is integrable i.e. its Nijenhuis tensor vanishes:* $N_J = 0$

Definition 3

A **Kähler manifold** *is a Riemannian manifold* M *equipped with a Kähler structure.*

Classification of 4-dim Thurston geometries

C.T.C. Wall, *Geometric structures on compact complex analytic surfaces*, Topology **25**, (1986), 119-152.

Kähler	complex non-Kähler	non-complex
$\mathbb{C}P^2$, $\mathbb{C}H^2$, $\mathbb{E}^4$ $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{S}^2 \times \mathbb{E}^2$, $\mathbb{S}^2 \times \mathbb{H}^2$ F^4, $\mathbb{E}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$	$\mathbb{S}^3 \times \mathbb{E}^1$, $\mathrm{Nil}_3 \times \mathbb{E}^1$ $\widetilde{\mathrm{SL}}_2\mathbb{R} \times \mathbb{E}^1$ Sol_0^4, Sol_1^4	$\mathbb{H}^4$, $\mathbb{S}^4$ $\mathbb{H}^3 \times \mathbb{E}^1$ Nil^4, $Sol^4_{m,n}$

Corollary 1.1

If X is one of $\mathbb{S}^3 \times \mathbb{E}^1$, $\mathrm{Nil}_3 \times \mathbb{E}^1$, $\widetilde{\mathrm{SL}}_2\mathbb{R} \times \mathbb{E}^1$, Sol_0^4, Sol_1^4, then X does not posses a Kähler structure compatible with the geometry.

complex non-Kahler $\Longrightarrow$ locally conformal Kähler (LCK)

LCK manifold

$M = (M, J, g)$ - Hermitian manifold with non-closed Kähler form Ω

Definition 4

M is said to be a *locally conformal Kähler manifold* (**LCK manifold**) if there exits an open covering $\{U_\alpha\}_{\alpha \in \Lambda}$ of M and a family of smooth functions $\sigma_\alpha : U_\alpha \to \mathbb{R}$ such that

$$d(e^{-\sigma_\alpha}\Omega) = 0 \quad \text{on} \quad U_\alpha \ \forall \alpha.$$

$U_\alpha = M \quad \Longrightarrow \quad M$ is *globally conformal Kähler (GCK) manifold*.

On LCK manifold 1-form $\omega = \mathbf{d}\sigma_\alpha$ (**Lee form**) is globally defined and satisfies

$$\mathbf{d\Omega} = \omega \wedge \mathbf{\Omega}.$$

The vector field B metrically dual to ω is called **Lee vector field**.

The vector field $A = JB$ is called **anti-Lee vector field**.

Magnetic curve

- in electromagnetic theory, a magnetic curve is a trajectory of charged particle moving in Euclidean space under a static magnetic field $\vec{B}$

Newton's second law of motion $\vec{F} = m\vec{a}$ implies

Lorentz force law

$$m\frac{d\vec{v}}{dt}(t) = q\ \vec{v}(t) \times \vec{B}_{\vec{r}(t)}$$

- *m - mass of the particle*
- *v - velocity of the particle*
- *q - charge of the particle*

Magnetic equation

$$\vec{\mathbb{B}} = (b_1, b_2, b_3) \longmapsto F = b_1 \; dx_2 \wedge dx_3 + b_2 \; dx_3 \wedge dx_1 + b_3 \; dx_1 \wedge dx_2$$

$$\text{Gauss's law for magnetism} \quad \nabla\vec{\mathbb{B}} = 0 \quad \Longleftrightarrow \quad dF = 0$$

- generalization to arbitrary Riemannian manifold with a closed 2-form F

Lorentz equation

$$\boxed{\nabla_{\gamma'}\gamma' = q\,\Phi\,(\gamma')}$$

- Φ - an endomorphism field $\rightarrow$ Lorentz force

$$g(\Phi X, Y) = F(X, Y)$$

Definition 5

A curve $\gamma(t)$ is called a **magnetic curve** if it satisfies the Lorentz equation.

Notice: $\Phi = 0 \implies \nabla_{\gamma'}\gamma' = 0 \implies$ geodesic equation

Magnetic curves in 3-dim Thurston geometries

- S. L. Druță-Romaniuc, J. Inoguchi, M. I. Munteanu, A. I. Nistor, Magnetic curves in Sasakian manifolds, *J. Nonlinear Math. Phys.* **22** (2015) 3, 428–447.
- S.L. Druță-Romaniuc, J. Inoguchi, M.I. Munteanu, A.I. Nistor, Magnetic curves in cosymplectic manifolds, *Report Math. Phys.* **78** (2016) 33–47.
- J. Inoguchi, M. I. Munteanu, Magnetic curves in the real special linear group, *Adv. Theor. Math. Phys.* **23** (2019) 8, 2161–2205.
- J. Inoguchi, M. I. Munteanu, A. I. Nistor, Magnetic curves in quasi-Sasakian 3-manifolds, *Anal. Math. Phys.* **9** (2019), 43–61.
- M.I. Munteanu, Magnetic curves in a Euclidean space: One example, several approaches, Publ. de L'Institut Math. **94** (108) (2013) 141–150.
- Z. Erjavec, J. Inoguchi, Magnetic curves in Sol_3, *J. Nonlinear Math. Phys.* **25** (2) (2018), 198–210.
- Z. Erjavec, J. Inoguchi, On magnetic curves in almost cosymplectic Sol space, *Results Math.* **75**:113 (2020) 16 pg.

Magnetic curves vs J-trajectories

- on a Kähler manifold: Kähler form $\Longrightarrow$ Kähler magnetic field
- on an LCK manifold: Kähler form is not closed (not magnetic!)

Definition 6

A curve $\gamma(t)$ is called a J-**trajectory** if it satisfies the equation $\nabla_{\dot\gamma}\dot\gamma = qJ\dot\gamma$.

Gauss's law in 4-dim

- on a Kähler manifold

$$d\Omega = 0 \quad \Longrightarrow \quad \omega = 0$$

- on an LCK manifold

$$d\Omega = \omega \wedge \Omega \quad \Longrightarrow \quad d\omega = 0$$

2. Geometry of Sol_0^4 space

- $\mathbb{R}^4(x, y, z, t)$ equipped with *Riemannian metric*

$$(ds)^2 = e^{-2t}\left((dx)^2 + (dy)^2\right) + e^{4t}(dz)^2 + (dt)^2$$

$$g_{ij} = \begin{pmatrix} e^{-2t} & 0 & 0 & 0 \\ 0 & e^{-2t} & 0 & 0 \\ 0 & 0 & e^{4t} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(x_1, y_1, z_1, t_1) * (x_2, y_2, z_2, t_2) = (x_1 + e^{t_1}x_2,\ y_1 + e^{t_1}y_2,\ z_1 + e^{-2t_1}z_2,\ t_1 + t_2)$$

- warped product representations of Sol_0^4:

$$\mathbb{H}^2(-4) \times_{e^{-t}} \mathbb{E}^2,$$

$$\mathbb{H}^3(-1) \times_{e^{2t}} \mathbb{E}^1.$$

Levi-Civita connection

The orthonormal frame $\{e_1, e_2, e_3, e_4\}$

$$\mathbf{e_1} = e^t \frac{\partial}{\partial x}, \quad \mathbf{e_2} = e^t \frac{\partial}{\partial y}, \quad \mathbf{e_3} = e^{-2t} \frac{\partial}{\partial z}, \quad \mathbf{e_4} = \frac{\partial}{\partial t}$$

The dual coframe $\{\theta^1, \theta^2, \theta^3, \theta^4\}$

$$\vartheta^{\mathbf{1}} = e^{-t} dx, \quad \vartheta^{\mathbf{2}} = e^{-t} dy, \quad \vartheta^{\mathbf{3}} = e^{2t} dz, \quad \vartheta^{\mathbf{4}} = dt.$$

Levi-Civita connection

$$\begin{array}{llll}
\nabla_{\mathbf{e_1}}\mathbf{e_1} = \mathbf{e_4} & \nabla_{\mathbf{e_1}}\mathbf{e_2} = 0 & \nabla_{\mathbf{e_1}}\mathbf{e_3} = 0 & \nabla_{\mathbf{e_1}}\mathbf{e_4} = -\mathbf{e_1} \\
\nabla_{\mathbf{e_2}}\mathbf{e_1} = 0 & \nabla_{\mathbf{e_2}}\mathbf{e_2} = \mathbf{e_4} & \nabla_{\mathbf{e_2}}\mathbf{e_3} = 0 & \nabla_{\mathbf{e_2}}\mathbf{e_4} = -\mathbf{e_2} \\
\nabla_{\mathbf{e_3}}\mathbf{e_1} = 0 & \nabla_{\mathbf{e_3}}\mathbf{e_2} = 0 & \nabla_{\mathbf{e_3}}\mathbf{e_3} = -2\mathbf{e_4} & \nabla_{\mathbf{e_3}}\mathbf{e_4} = 2\mathbf{e_3} \\
\nabla_{\mathbf{e_4}}\mathbf{e_1} = 0 & \nabla_{\mathbf{e_4}}\mathbf{e_2} = 0 & \nabla_{\mathbf{e_4}}\mathbf{e_3} = 0 & \nabla_{\mathbf{e_4}}\mathbf{e_4} = 0
\end{array}$$

Hermitian structure on Sol_0^4

g-orthogonal almost complex structure J

$$Je_1 = -e_2, \quad Je_2 = e_1, \quad Je_3 = e_4, \quad Je_4 = -e_3.$$

Kähler form

$$\mathbf{\Omega = 2e^{-2t}dx \wedge dy - 2e^{2t}dz \wedge dt}$$

$$d\Omega = \omega \wedge \Omega \quad \Longrightarrow \quad \omega = \mathbf{-2dt}$$

The *homogeneous Hermitian space* (Sol_0^4, J) is a (*non-Kähler*) globally conformal Kähler surface with Lee field $\mathbf{B = -2e_4}$ and anti Lee field $\mathbf{A = 2e_3}$.

Typical submanifolds of Sol_0^4

$$g = e^{-2t}\left((dx)^2 + (dy)^2\right) + e^{4t}(dz)^2 + (dt)^2$$

Euclidean plane

$$M(1,2;z_0,t_0) := \{(x,y,z_0,t_0) \in \mathrm{Sol}_0^4\}$$

- non totally geodesic in Sol_0^4
- a fiber of $\mathrm{Sol}_0^4 = \mathbb{H}^2(-4) \times_{e^{-t}} \mathbb{E}^2$
- totally umbilic in Sol_0^4

Hyperbolic plane

$$M(3,4;x_0,y_0) := \{(x_0,y_0,z,t) \in \mathrm{Sol}_0^4\}$$

- totally geodesic in Sol_0^4
- a leaf of $\mathrm{Sol}_0^4 = \mathbb{H}^2(-4) \times_{e^{-t}} \mathbb{E}^2$

Typical submanifolds of Sol_0^4

$$g = e^{-2t}\left((dx)^2 + (dy)^2\right) + e^{4t}(dz)^2 + (dt)^2$$

Euclidean 3-space

$$M(1,2,3;t_0) := \{(x,y,z,t_0) \in \mathrm{Sol}_0^4\}$$

- minimal in Sol_0^4
- non totally geodesic in Sol_0^4

Hyperbolic 3-space

$$M(1,2,4;z_0) := \{(x,y,z_0,t) \in \mathrm{Sol}_0^4\}$$

- totally geodesic in Sol_0^4
- a leaf of $\mathrm{Sol}_0^4 = \mathbb{H}^3(-1) \times_{e^{2t}} \mathbb{E}^1$.

3. J-trajectories in Sol_0^4 space

$$\gamma(\mathbf{s}) = (\mathbf{x}(\mathbf{s}), \mathbf{y}(\mathbf{s}), \mathbf{z}(\mathbf{s}), \mathbf{t}(\mathbf{s})) \implies \dot\gamma(s) = \dot x(s)\frac{\partial}{\partial x} + \dot y(s)\frac{\partial}{\partial y} + \dot z(s)\frac{\partial}{\partial z} + \dot t(s)\frac{\partial}{\partial t}$$

$$\dot\gamma(s) = e^{-t(s)}\dot x(s)e_1 + e^{-t(s)}\dot y(s)e_2 + e^{2t(s)}\dot z(s)e_3 + \dot t(s)e_4$$

$$\boxed{\nabla_{\dot\gamma}\dot\gamma = q\,J\,\dot\gamma}$$

$$\begin{aligned}\nabla_{\dot\gamma}\dot\gamma = \;& e^{-t(s)}\left(\ddot x(s) - 2\dot x(s)\dot t(s)\right)e_1\\ &+ e^{-t(s)}\left(\ddot y(s) - 2\dot y(s)\dot t(s)\right)e_2\\ &+ e^{2t(s)}\left(\ddot z(s) + 4\dot z(s)\dot t(s)\right)e_3\\ &+ \left(\ddot t(s) + e^{-2t(s)}\left(\dot x(s)^2 + \dot y(s)^2\right) - 2e^{4t(s)}\dot z(s)^2\right)e_4\end{aligned}$$

$$J\dot\gamma(s) = e^{-t(s)}\dot y(s)e_1 - e^{-t(s)}\dot x(s)e_2 - \dot t(s)e_3 + e^{2t(s)}\dot z(s)e_4$$

The system

System of differential equations

$$
\begin{aligned}
\ddot{x}(s) - 2\dot{x}(s)\dot{t}(s) &= q\ \dot{y}(s) \\
\ddot{y}(s) - 2\dot{y}(s)\dot{t}(s) &= -q\ \dot{x}(s) \\
\ddot{z}(s) + 4\dot{z}(s)\dot{t}(s) &= -q\ e^{-2t(s)}\dot{t}(s) \\
\ddot{t}(s) + e^{-2t(s)}\left(\dot{x}(s)^2 + \dot{y}(s)^2\right) &= e^{2t(s)}\left(q\dot{z}(s) + 2e^{2t(s)}\dot{z}(s)^2\right)
\end{aligned} \tag{1}
$$

Arc length condition

$$
e^{-2t(s)}\dot{x}(s)^2 + e^{-2t(s)}\dot{y}(s)^2 + e^{4t(s)}\dot{z}(s)^2 + \dot{t}(s)^2 = 1
$$

Solving the system

$$\dot{x}(s) = ae^{2t(s)}\sin(qs+c), \quad \dot{y}(s) = ae^{2t(s)}\cos(qs+c)$$

$$\dot{z}(s) = be^{-4t(s)} - \frac{q}{2}e^{-2t(s)}, \quad a,b,c \in \mathbb{R}$$

$$\ddot{t}(s) + a^2e^{2t(s)} + bqe^{-2t(s)} - 2b^2e^{-4t(s)} = 0$$

Arc length condition

$$\dot{t}(s)^2 + a^2e^{2t(s)} - bqe^{-2t(s)} + b^2e^{-4t(s)} + \frac{q^2}{4} - 1 = 0$$

- Case 1: $a = b = 0$ and $\dot{t}(s)^2 = 1 - \frac{q^2}{4}$
- Case 2: $t(s) = \text{const} = k,\ a = a(k,q)$ and $b = b(k,q)$
- Case 3: $a = 0$ and $b \neq 0$
- Case 4: $a \neq 0$ and $b = 0$

Solution in Case 1

$\boxed{a = b = 0}$ $\quad \dot{t}(s)^2 = 1 - \frac{q^2}{4}$

$$x(s) = x_0, \qquad z(s) = \frac{q}{2\sqrt{4-q^2}} e^{-\sqrt{4-q^2}s - 2t_0},$$

$$y(s) = y_0, \qquad t(s) = \frac{\sqrt{4-q^2}}{2} s + t_0.$$

- J-trajectory lies in the hyperbolic plane $M(3, 4; x_0, y_0)$

Applying the coordinate change $X(s) := 2z(s),\ Y(s) := e^{-2t(s)}$,

$(X(s), Y(s))$ is a curve in $\quad \mathbb{H}^2(-4) = \{(X, Y) \in \mathbb{R}^2 \mid Y > 0\}$ given by

$$X = \frac{q}{\sqrt{4-q^2}} Y$$

J-trajectory in Case 1

J-trajectories for $x(t) = x_0,\ y(t) = 0,\ t_0 = 0,\ q = 1,\ s \in [-3, 3]$

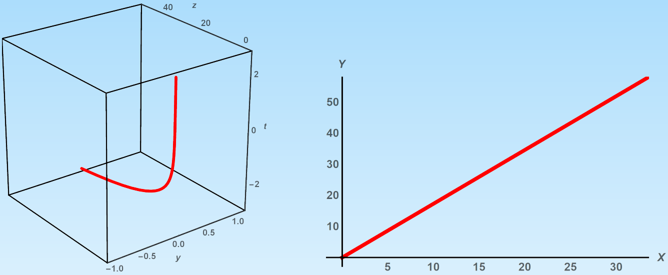

Figure: J-trajectories in $M(3, 4; x_0, 0)$ and $\mathbb{H}^2(-4)$

- curvatures: $\kappa_1 = |q|$ and $\kappa_2 = 0$

Solution in Case 2

$\boxed{t(s) = t_0 = const} \Longrightarrow a, b = const$

$$a = \mp \frac{\sqrt{2}e^{-t_0}}{6}\sqrt{12 - q^2 - q\sqrt{q^2+12}} \qquad b = \frac{e^{2t_0}}{6}(2q - \sqrt{q^2+12}),$$
$$a = \mp \frac{\sqrt{2}e^{-t_0}}{6}\sqrt{12 - q^2 + q\sqrt{q^2+12}} \qquad b = \frac{e^{2t_0}}{6}(2q + \sqrt{q^2+12})$$

$$x(s) = -\frac{a}{q}e^{2t_0}\cos(qs+c), \qquad z(s) = e^{-2t_0}\left(be^{-2t_0} - \frac{q}{2}\right)s + d,$$
$$y(s) = \frac{a}{q}e^{2t_0}\sin(qs+c), \qquad t(s) = t_0.$$

- J-trajectory lies in the Euclidean space $M(1,2,3;t_0)$

J-trajectory in Case 2

J-trajectory for $q = 1$, $a = 2$, $b = 3$, $c = 0$, $d = 0$, $t_0 = 1$, and $s \in [-10, 10]$.

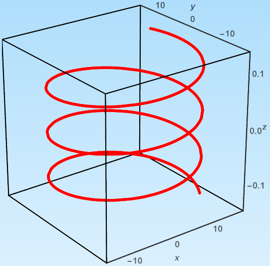

Figure: J-trajectory in $M(1, 2, 3; t_0)$

- curvatures: $\kappa_1 = |q|$ and $\kappa_2 = \frac{1}{4}\sqrt{1 - \left(be^{-2t_0} - \frac{q}{2}\right)^2}$

Solution in Case 3

$a = 0 \quad \text{and} \quad b \neq 0$

$$x(s) = x_0, \qquad z(s) = \int be^{-4t(s)} - \frac{q}{2} e^{-2t(s)} ds,$$
$$y(s) = y_0, \qquad t(s) \text{ is a solution of (2).}$$

$$\text{Arctan}^2 \left[\frac{e^{2t(s)}(q^2-4) - 2bq}{\sqrt{q^2-4}\sqrt{4e^{4t(s)} - (qe^{2t(s)} - 2b)^2}} \right] = (q^2-4)(c_1 - s)^2, \quad c_1 \in \mathbb{R} \tag{2}$$

$q = 2$

$$x(s) = x_0, \qquad z(s) = \frac{2(s-c_2)}{b\left(1 + 4(s-c_2)^2\right)},$$
$$y(s) = y_0, \qquad t(s) = \frac{1}{2} \ln \left[\frac{b}{2} \left(1 + 4(s-c_2)^2\right) \right].$$

- J-trajectory lies in the hyperbolic plane $M(3, 4; x_0, y_0)$

$$X(s) := 2z(s), \ Y(s) := e^{-2t(s)} \quad \Longrightarrow \quad X^2 = \frac{2 - bY}{16b^2}$$

J-trajectory in Case 3

J-trajectories for $\boxed{q=2}$ $x(t)=x_0,\ y(t)=0,\ b=2,\ c_2=0,001,\ s\in[-3,3]$

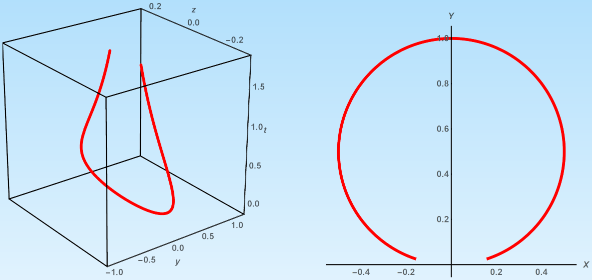

Figure: J-trajectories in $M(3,4;x_0,0)$ and $\mathbb{H}^2(-4)$

- curvatures: $\kappa_1=|q|$ and $\kappa_2=0$

Solution in Case 4

$a \neq 0 \quad \text{and} \quad b = 0$

$$\dot{t}(s)^2 + a^2 e^{2t(s)} + \frac{q^2}{4} - 1 = 0$$

$q \in \langle -2, 2 \rangle$

$$t(s) = \frac{1}{2} \ln \left[\frac{4 - q^2}{4a^2} \operatorname{sech}^2 \left(\frac{1}{2} \sqrt{4 - q^2} \, (\pm s - d) \right) \right] \tag{3}$$

$$x(s) = \int a e^{2t(s)} \sin(qs + c) ds, \qquad z(s) = \int -\frac{q}{2} e^{-2t(s)} ds,$$

$$y(s) = \int a e^{2t(s)} \cos(qs + c) ds, \qquad t(s) \text{ is given by (3).}$$

The main theorem

Theorem 3.1

The J-trajectories in the model space Sol_0^4 are solutions of the ODE-system (1)*. In particular, some analytical solutions of* (1) *are*

(a) *curves given by parametric equations*

$$x(s) = x_0, \qquad z(s) = \mp e^{-2t_0} s + z_0,$$
$$y(s) = y_0, \qquad t(s) = t_0, \qquad \text{for } q = \pm 2,$$

or

$$x(s) = x_0, \qquad z(s) = \frac{q}{2\sqrt{4-q^2}} e^{-\sqrt{4-q^2}s - 2t_0},$$
$$y(s) = y_0, \qquad t(s) = \frac{\sqrt{4-q^2}}{2} s + t_0, \qquad \text{for } q \in \langle -2, 2 \rangle,$$

where $x_0, y_0, z_0, t_0 \in \mathbb{R}$,

(b) *curves given by parametric equations*

$$x(s) = -\tfrac{a}{q} \cos(qs + c), \qquad z(s) = e^{-2k} \left(be^{-2k} - \tfrac{q}{2} \right) s + d,$$
$$y(s) = \tfrac{a}{q} \sin(qs + c), \qquad t(s) = k,$$

where a, b *are given by* (2)*,* c*,* d*,* $k \in \mathbb{R}$ *and* $q \in \mathbb{R}\backslash\{0\}$,

The main theorem

Theorem 3.2

(c) *curves given by parametric equations*

$$x(s) = x_0, \qquad z(s) = \int be^{-4t(s)} - \tfrac{q}{2} e^{-2t(s)} ds,$$

$$y(s) = y_0, \qquad t(s) \text{ is a solution of (2)},$$

where $x_0, y_0, b, q \in \mathbb{R}$,

(d) *curves given by parametric equations*

$$x(s) = \int ae^{2t(s)} \sin(qs + c)\, ds, \qquad z(s) = \frac{a^2 q}{q^2 - 4} \left((s \pm d) + \frac{1}{\sqrt{4-q^2}} \sinh(\sqrt{4 - q^2}(s \pm d)) \right),$$

$$y(s) = \int ae^{2t(s)} \cos(qs + c)\, ds, \qquad t(s) = \frac{1}{2} \ln \left[\frac{4-q^2}{4a^2} \operatorname{sech}^2 \left(\frac{1}{2} \sqrt{4 - q^2}\, (\pm s - d) \right) \right],$$

where $q \in \langle -2, 2 \rangle$ *and* $a, c, d \in \mathbb{R}$.

4. Curvature properties of J-trajectories

Definition 7

If γ is a curve in a Riemannian manifold M, parametrized by arc length s, we say that γ is a **Frenet curve of osculating order r** if there exist orthonormal vector fields E_1, $E_2, \cdots, E_r$ along γ such that

$$\dot{\gamma} = E_1,\ \nabla_{\dot{\gamma}} E_1 = \kappa_1 E_2,\ \nabla_{\dot{\gamma}e} E_2 = -\kappa_1 E_1 + \kappa_2 E_3, \cdots,$$
$$\nabla_{\dot{\gamma}} E_{r-1} = -\kappa_{r-2} E_{r-2} + \kappa_{r-1} E_r,\ \nabla_{\dot{\gamma}} E_r = -\kappa_{r-1} E_{r-1},$$

where κ_1, $\kappa_2, \cdots, \kappa_{r-1}$ are positive C^∞ functions of s.

- A **geodesic** is regarded as a Frenet curve of osculating order 1.
- A **circle** is defined as a Frenet curve of osculating order 2 with *constant* κ_1.
- A **helix** of order r is a Frenet curve of osculating order r, such that all the curvatures $\kappa_1, \kappa_2, \cdots, \kappa_{r-1}$ are constant.

On curvatures of J-trajectories in Sol_0^4

Proposition 4.1

Let γ be a non-geodesic J-trajectory with strength $q \neq 0$ parameterized by arc length in Sol_0^4. **Then $\kappa_2 = 0$ if and only if both x-coordinate and y-coordinate of γ are constant**.

Proposition 4.2

Let γ be a non-geodesic J-trajectory with strength $q \neq 0$ parameterized by arc length in Sol_0^4. Assume that $\kappa_2 > 0$. **Then κ_2 is a constant if and only if t-coordinate of γ is a constant**.

Proposition 4.3

Let γ be a non-geodesic J-trajectory with strength $q \neq 0$ parameterized by arc length in Sol_0^4. If we assume $\kappa_2 > 0$, **then $\kappa_3 = 0$ if and only if z is a constant.**

Summary

- *Some basic definitions and facts (complex structure, Kähler form, LCK manifold) are repeated.*
- *Geometry of Sol_0^4 space is described.*
- *J-trajectories in Sol_0^4 space are examined.*
- *Curvature properties of J-trajectories in Sol_0^4 are considered.*

Recent work

Z. Erjavec, J. Inoguchi, Magnetic curves in $\mathbb{H}^3 \times \mathbb{R}$, accepted for publication in J Korean Math Soc

Z. Erjavec, J. Inoguchi, J-trajectories in 4-dimensional solvable Lie group Sol_0^4, submitted.

Z. Erjavec, J. Inoguchi, J-trajectories in 4-dimensional solvable Lie group Sol_1^4, in prepair.

Thank you for your attention!

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