Strongly regular configurations

Vedran Krčadinac*

University of Zagreb, Croatia

(joint work with Marién Abreu, Martin Funk, and Domenico Labbate)

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A graph is called strongly regular with parameters $SRG(n, d, \lambda, \mu)$ if it has n vertices, is regular of degree d, and every two vertices have λ common neighbors if they are adjacent, and μ common neighbors if they are not adjacent.

Strongly regular configurations

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The point graph is a

$$SRG\left(\frac{(s+1)(st+\alpha)}{\alpha},\ s(t+1),\ s-1+t(\alpha-1),\ \alpha(t+1)\right),$$

and the line graph is a

$$SRG\left(\frac{(t+1)(st+\alpha)}{\alpha},\ t(s+1),\ t-1+s(\alpha-1),\ \alpha(s+1)\right).$$

Partial geometries

Partial geometries include Steiner 2-designs $pg(s,t,s+1)$ and their duals $pg(s,t,t+1)$, Bruck nets $pg(s,t,t)$ and their duals $pg(s,t,s)$ (transversal designs), and generalized quadrangles $pg(s,t,1)$ as special cases.

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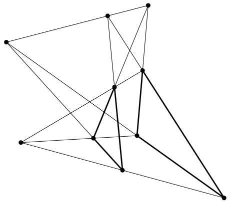
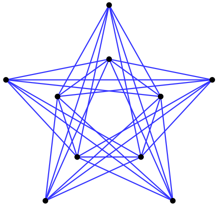
There are configurations with both associated graphs strongly regular that are **not** partial geometries – e.g. the Desargues configuration (10_3):



$\rightsquigarrow$



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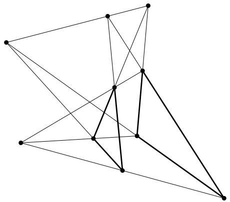

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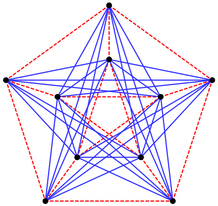

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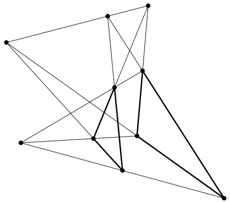

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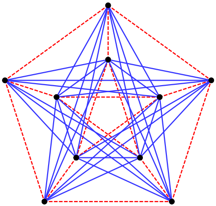

$SRG(10,6,3,4)$ (complement of the Petersen graph)

The Desargues configuration is a **semipartial geometry** $spg(2,2,2,4)$.

Semipartial geometries

I. Debroey, J. A. Thas, *On semipartial geometries*, J. Comb. Theory A **25** (1978), 242–250.

A semipartial geometry $spg(s, t, \alpha, \mu)$ is a configuration with $k = s + 1$ and $r = t + 1$ such that for every non-incident point-line pair (P, ℓ), there are either 0 or α points on ℓ collinear with P. Furthermore, for every pair of non-collinear points, there are exactly μ points collinear with both.

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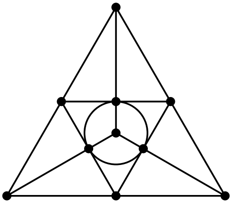
The point graph is a

$$SRG\left(1 + \frac{s(t+1)(\mu + t(s+1-\alpha))}{\mu}, s(t+1), s-1+t(\alpha-1), \mu\right).$$

The line graph need not be strongly regular. However, in the symmetric case ($v = b$ or $k = r$ or $s = t$) the line graph is also strongly regular with the same parameters.

Other examples of such configurations

Another (10_3) configuration:



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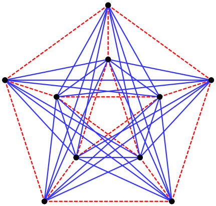

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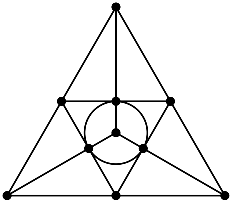

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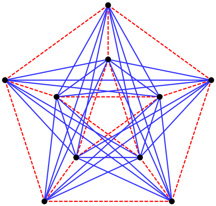

SRG$(10, 6, 3, 4)$
(complement of the Petersen graph)

This configuration is not a semipartial geometry and does not belong to other known generalizations of partial geometries such as strongly regular (α, β)-geometries:

N. Hamilton, R. Mathon, *Strongly regular* (α, β)*-geometries*, J. Combin. Theory Ser. A **95** (2001), no. 2, 234–250.

Non-symmetric examples?

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Theorem.

If the point graph of a (v_r, b_k) configuration is strongly regular, then the configuration is a partial geometry or $v \leq b$.

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Corollary.

If both associated graphs of a (v_r, b_k) configuration are strongly regular, then the configuration is a partial geometry or $v = b$.

Definitions

Definition.

A strongly regular configuration with parameters $(v_k; \lambda, \mu)$ is a symmetric (v_k) configuration with the point graph a $SRG(v, k(k-1), \lambda, \mu)$.

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Proposition.

A strongly regular $(v_k; \lambda, \mu)$ configuration that is not a projective plane is proper if and only if $(v-k)(\lambda+1) > k(k-1)^3$ holds.

Definitions

Projective planes of order n are partial geometries $pg(n, n, n+1)$ and satisfy equality $(v-k)(\lambda+1) = k(k-1)^3$, but have regular incidence matrices. The associated point and line graphs are complete.

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Second case: $\mu = k(k-1) \iff$ the graphs are complete multipartite
$\iff$ non-collinearity of points is an equivalence relation
$\iff$ the configuration is an elliptic semiplane.

P. Dembowski, *Finite geometries*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 44, Springer-Verlag, 1968.

Strongly regular configurations with spg parameters

We focus on strongly regular configurations that are **proper** and **primitive**, i.e. such that neither collinearity nor non-collinearity of points are equivalence relations. This is equivalent with $0 < \mu < k(k-1)$.

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A. J. Hoffman, R. R. Singleton, *On Moore graphs with diameters* 2 *and* 3, IBM J. Res. Develop. **4** (1960), 497–504.

Moore graphs have parameters $SRG(k^2+1, k, 0, 1)$ with $k \in \{2, 3, 7, 57\}$.

$k = 2 \rightsquigarrow$ the pentagon
$k = 3 \rightsquigarrow$ the Petersen graph
$k = 7 \rightsquigarrow$ the Hoffman-Singleton graph
$k = 57 \rightsquigarrow$ **?**

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- points are vertices of a Moore graph $SRG(k^2+1, k, 0, 1)$,
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$\rightsquigarrow$ semipartial geometry $spg(k-1,k-1,k-1,(k-1)^2)$

strongly regular $((k^2+1)_k;k(k-2),(k-1)^2)$ configuration

The point graph is the complementary $SRG(k^2+1,k(k-1),k(k-2),(k-1)^2)$.

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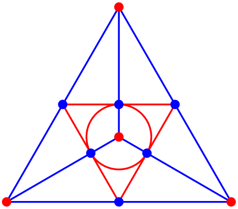

There is another $(10_3; 3, 4)$ configuration which is **not** a semipartial geometry!

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$k = 7 \rightsquigarrow$ semipartial geometry $spg(6,6,6,36)$
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$$v = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q, \quad b = \begin{bmatrix} n+1 \\ 3 \end{bmatrix}_q, \quad r = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q, \quad k = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q$$

$\rightsquigarrow$ semipartial geometry $spg(k-1, r-1, q+1, (q+1)^2)$

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$LP(4,q) \rightsquigarrow$ semipartial geometry $spg(q(q+1), q(q+1), q+1, (q+1)^2)$
strongly regular $(v_k; \lambda, \mu)$ configuration for

$$v = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q, \quad k = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q, \quad \lambda = q^3 + 2q^2 + q - 1, \quad \mu = (q+1)^2$$

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Theorem.

For every prime power q, there are at least four strongly regular $(v_k;\lambda,\mu)$ configuration with these parameters. One of them is the semipartial geometry $LP(4,q)$ and the others are not semipartial geometries.

Polarity transformations

Proof. Polarity transformations of $LP(n, q) \ldots$

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Modify incidence of the POINTS and LINES of $LP(n,q)$ (i.e. lines and planes of $PG(4,q)$) contained in H_0, or containing P_0:

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Similar transformations:

D. Jungnickel, V. D. Tonchev, *Polarities, quasi-symmetric designs, and Hamada's conjecture*, Des. Codes Cryptogr. **51** (2009), no. 2, 131–140.

Strongly regular configurations with non-spg parameters

Are there strongly regular configurations with parameters different from semipartial geometries?

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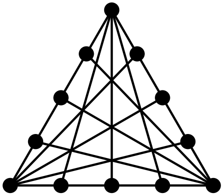
Are there strongly regular configurations with parameters different from semipartial geometries?

Theorem.

Let $\mathcal{P}$ be a projective plane of order $n \geq 5$ and A, B, C be three non-collinear points. By deleting all points on the lines AB, AC, BC and all lines through the points A, B, C, there remains a strongly regular $(v_k; \lambda, \mu)$ configuration with $v = (n-1)^2$, $k = n-2$, $\lambda = (n-4)^2 + 1$, and $\mu = (n-3)(n-4)$. The configuration is not a (semi)partial geometry.



Strongly regular configurations with non-spg parameters

Example: planes of order $n = 9 \rightsquigarrow (64_7; 26, 30)$ configurations

$PG(2,9)$	$\rightsquigarrow$	1 configuration
Hall	$\rightsquigarrow$	6 configurations
Dual Hall	$\rightsquigarrow$	6 configurations (duals)
Hughes	$\rightsquigarrow$	16 configurations

Total: **29** non-isomorphic $(64_7; 26, 30)$ configurations

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Hypothesis: every $(v_k; \lambda, \mu)$ configuration with $v = (n-1)^2$, $k = n-2$, $\lambda = (n-4)^2 + 1$, $\mu = (n-3)(n-4)$ can be uniquely embedded in a projective plane of order n.

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Sporadic examples:

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- POINTS are the points of $GH(2, 2)$,
- LINES are sets of 6 points collinear with a given point of $GH(2, 2)$.

This is a $(63_6; 13, 15)$ configuration with full automorphism group $PSU(3, 3) : \mathbb{Z}_2$ of order 12096. Another such configuration is obtained from the dual of $GH(2, 2)$.

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More examples constructed from difference sets: the next talk *Configurations from strong deficient difference sets* by **Marién Abreu**.

A table of feasible parameters

Feasible parameters $(v_k; \lambda, \mu)$:

- $0 < \mu < k(k-1)$ holds, i.e. the configuration is *primitive*,
- the corresponding $SRG(v, k(k-1), \lambda, \mu)$ exist or cannot be ruled out,
- the parameters satisfy the following necessary conditions:

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Theorem (Brouwer–Haemers–Tonchev, 1996).

If a strongly regular $(v_k; \lambda, \mu)$ configuration exists, then $(r+k)^f (s+k)^g$ is the square of an integer, where r, s, f, g are given by

$$r, s = \frac{1}{2}\left(\lambda - \mu \pm \sqrt{(\lambda-\mu)^2 - 4(\mu - k(k-1))}\right),$$

$$f, g = \frac{1}{2}\left(v - 1 \mp \frac{(r+s)(v-1) + 2k(k-1)}{r-s}\right).$$

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If a strongly regular $(v_k; \lambda, \mu)$ configuration exists, then

$$(v-k)(\lambda+1) \geq k(k-1)^3.$$

Equality holds if and only if the configuration is a partial geometry.

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Using Brouwer's table of strongly regular graphs, we made the following table of feasible parameters for strongly regular configurations with $v \leq 200$.

A. E. Brouwer, *Parameters of strongly regular graphs.*
https://www.win.tue.nl/~aeb/graphs/srg/srgtab.html

A table of feasible parameters

No.	$(v_k; \lambda, \mu)$	#Cf	#SCf
1	$(10_3; 3, 4)$	**2**	**2**
2	$(13_3; 2, 3)$	**1**	**1**
3	$(16_3; 2, 2)$	**1**	**1**
4	$(25_4; 5, 6)$	**0**	**0**
5	$(36_5; 10, 12)$	**1**	**1**
6	$(41_5; 9, 10)$	?	?
7	$(45_4; 3, 3)$	**0**	**0**
8	$(49_4; 5, 2)$	**0**	**0**
9	$(49_6; 17, 20)$	1	1
10	$(50_7; 35, 36)$	211	111
11	$(61_6; 14, 15)$	?	?

A table of feasible parameters

No.	$(v_k; \lambda, \mu)$	#Cf	#SCf
12	$(63_6; 13, 15)$	4	2
13	$(64_7; 26, 30)$	29	11
14	$(81_8; 37, 42)$	?	?
15	$(85_6; 11, 10)$	?	?
16	$(85_7; 20, 21)$	?	?
17	$(96_5; 4, 4)$	1	1
18	$(99_7; 21, 15)$	?	?
19	$(100_9; 50, 56)$	1	1
20	$(105_9; 51, 45)$	?	?
21	$(113_8; 27, 28)$	?	?
22	$(120_8; 28, 24)$	1	1

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No.	$(v_k; \lambda, \mu)$	#Cf	#SCf
23	$(121_5; 9, 2)$	**0**	**0**
24	$(121_6; 11, 6)$	?	?
25	$(121_9; 43, 42)$	?	?
26	$(121_{10}; 65, 72)$	?	?
27	$(125_9; 45, 36)$	?	?
28	$(136_6; 15, 4)$	?	?
29	$(136_9; 36, 40)$	?	?
30	$(144_{11}; 82, 90)$	1	1
31	$(145_9; 35, 36)$	?	?
32	$(153_8; 19, 21)$	?	?
33	$(155_7; 17, 9)$	4	2

A table of feasible parameters

No.	$(v_k; \lambda, \mu)$	#Cf	#SCf
34	$(169_9; 31, 30)$	?	?
35	$(169_{12}; 101, 110)$	?	?
36	$(171_{11}; 73, 66)$	?	?
37	$(175_6; 5, 5)$	?	?
38	$(181_{10}; 44, 45)$	?	?
39	$(196_{10}; 40, 42)$	?	?
40	$(196_{13}; 122, 132)$	?	?
41	$(196_{13}; 125, 120)$	?	?

A non-existence result

The $n \times n$ rook graph:

- vertices are pairs (x, y) with $x, y \in \{1, \ldots, n\}$,
- (x_1, y_1) and (x_2, y_2) are adjacent if $x_1 = x_2$ or $y_1 = y_2$.

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Theorem (Shrikhande, 1959).

For $n > 4$, the only $SRG(n^2, 2(n-1), n-2, 2)$ is the $n \times n$ rook graph. For $n = 4$, there are two such graphs.

S. S. Shrikhande, *The uniqueness of the L_2 association scheme*, Ann. Math. Statist. **30** (1959), 781–798.

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Theorem.

The $n \times n$ rook graph cannot be the point graph of a strongly regular configuration.

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Corollary.

Strongly regular $(v_k; \lambda, \mu)$ configurations with the following feasible parameters do not exist for $k > 3$:

$$v = \left(\binom{k}{2} + 1\right)^2, \quad \lambda = \binom{k}{2} - 1, \quad \mu = 2.$$

The End

Thanks for your attention!