Eigenvalues and $[a, b]$-factors in Regular Graphs

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Question

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- Let g and f be integer-valued functions on $V(G)$ such that for every vertex $v \in V(G)$, $0 \leq g(v) \leq f(v) \leq d_G(v)$. A (g, f)-factor of G is a spanning subgraph F of G such that for each vertex $v \in V(G)$, $g(v) \leq d_F(v) \leq f(v)$.

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- For nonnegative integers a and b with $a \leq b$, an (even or odd) $[a, b]$-factor of G is a (g, f)-factor F such that ($d_F(v)$ is even or odd and) $g(v) = a$ and $f(v) = b$ for all $v \in V(G)$; if $a = b = k$, then we call it a k-factor.

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- A graph G is h-edge-connected if for $S \subseteq E(G)$ with $|S| < h$, $G - S$ is connected; the edge-connectivity of G, denoted $\kappa'(G)$, is the maximum h such that G is h-edge-connected.

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- A graph G is h-edge-connected if for $S \subseteq E(G)$ with $|S| < h$, $G - S$ is connected; the edge-connectivity of G, denoted $\kappa'(G)$, is the maximum h such that G is h-edge-connected.
- A graph G is r-regular if every vertex has the same degree r.

Characterization of Triples (k, r, h)

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Theorem (Gallai, 1950)

Let k, r, and h be positive integers with $1 \leq r$, and let G be a h-edge-connected r-regular graph with n vertices. If one of the following condition holds, then G contains a k-factor.
(i) r is even, k is odd, n is even, and $r \leq h \min\{k, r-k\}$;
(ii) r is odd, k is even, and $r \leq (r-k)h$;
(iii) both r and k are odd and $r \leq kh$.

Characterization of Triples (k, r, h)

Theorem (Bollobas, Saito, and Wormald, 1985)

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(i) r is even, k is odd, n is even, and $r \leq h \min\{k, r-k\}$;

(ii) r is odd, k is even, and $r \leq (r-k)h^*$, where $h^* \in \{h, h+1\}$ is an odd number;

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Niessen and Randerath (1998) extended their result by adding one more condition in terms of the number of vertices. In fact, they proved an upper bound for the number of vertices related to k, r, h to guarantee the existences of a k-factor in an n-vertex h-edge-connected r-regular graph.

Basic Definitions

- The adjacency matrix $A(G)$ (or simply A) of a graph G with vertex set $\{v_1, \ldots, v_n\}$ is the n-by-n matrix such that A_{ij} is the number of edges between v_i and v_j.

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- A matching in G is a set of disjoint edges.
- The matching number of G, written $\alpha'(G)$, is the maximum size of a matching in it.

λ_3 and 1-factor in Regular Graphs

Theorem (Brouwer and Haemers, 2005)

If G is a connected r-regular graph on even n vertices with

$$\lambda_3 \le \begin{cases} r - 1 + \frac{3}{r+1} & \text{if } r \text{ is even,} \\ r - 1 + \frac{3}{r+2} & \text{if } r \text{ is odd,} \end{cases}$$

then G has an 1-factor.

λ_3 and Near Perfect Matching in Regular Graphs

Theorem (Cioaba, Gregory, and Haemers, 2009)

If G is a connected r-regular graph of order n such that

$$\lambda_3 < \rho(r),$$

then $\alpha'(G) = \lfloor \frac{n}{2} \rfloor$.

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Theorem (Cioaba, Gregory and Haemers, 2009)

If θ is the largest root of $x^3 - x^2 - 6x + 2 = 0$, then

$$\rho(r) = \begin{cases} \theta = 2.85577... & \text{if } r = 3, \\ \frac{1}{2}(r - 2 + \sqrt{r^2 + 12}) & \text{if } r \geq 4 \text{ is even,} \\ \frac{1}{2}(r - 3 + \sqrt{(r+1)^2 + 16}) & \text{if } r \geq 5 \text{ is odd.} \end{cases}$$

Eigenvalues and Matching Number

Theorem (Cioaba and O, 2010)

Let $p \geq 3$ be an integer. If G is a h-edge-connected r-regular graph such that $\lambda_p(G) < \rho(r)$, then

$$\alpha'(G) > \begin{cases} \frac{n-p+\lfloor \frac{hp}{r} \rfloor}{2} & \text{when } r \equiv h \pmod 2 \\ \frac{n-p+\lfloor \frac{(h+1)p}{r} \rfloor}{2} & \text{when } r \equiv h+1 \pmod 2. \end{cases}$$

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Corollary (Cioaba, Gregory, and Haemaers, 2009)

If G is a connected r-regular graph of order n such that $\lambda_3 < \rho(r)$, then $\alpha'(G) > \frac{n-2}{2}$.

λ_3 and k-factor in Regular Graph

Theorem (Lu, 2010)

Let G be a connected r-regular graph on n vertices and $1 \le k < r$.
(i) r is even, k is odd, n is even, and m is an integer such that $r \le km$ and $r \le (r-k)m$. If

$$\lambda_3(G) < \begin{cases} \frac{1}{2}(r - 2 + \sqrt{(r+2)^2 - 4(m-2)}), & \text{if } m \text{ is even,} \\ \frac{1}{2}(r - 2 + \sqrt{(r+2)^2 - 4(m-1)}), & \text{if } m \text{ is odd,} \end{cases}$$

then G has a k-factor.

λ_3 and k-factor in Regular Graph

Theorem (Lu, 2010)

Let G be a connected r-regular graph on n vertices and $1 \leq k < r$. (ii) r is odd, k is even, and m is odd with $r \leq (r-k)m$ or even with $r \leq (r-k)(m+1)$. If

$$\lambda_3(G) < \begin{cases} \frac{1}{2}(r - 3 + \sqrt{(r+3)^2 - 4(m-1)}), & \text{if } m \text{ is even,} \\ \frac{1}{2}(r - 3 + \sqrt{(r+3)^2 - 4(m-1)}), & \text{if } m \text{ is odd,} \end{cases}$$

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Theorem (Lu, 2010)

Let G be a connected r-regular graph on n vertices and $1 \leq k < r$. (iii) both r and k are odd, and m is odd with $r \leq km$ or even with $r \leq k(m+1)$. If

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then G has a k-factor.

Eigenvalues and k-factor in h-edge-connected Regular Graph

Theorem (Gu, 2014)

Let r, k, h, h', and h^* be be positive integers such that $r \geq 3$, $k < r$, $h \leq r$, $h' \in \{h, h+1\}$ is an even number, and $h^* \in \{h, h+1\}$ is an odd number. Suppose that G is an h-edge-connected r-regular graph.

(i) For even r, odd k, even $|V(G)|$, $\hat{k} = \min\{k, r-k\}$, and $m = \lceil \frac{r}{\hat{k}} \rceil$, if $r \leq \hat{k}h'$, or if $r > \hat{k}h'$ and $\lambda_{\lceil \frac{2r}{r-\hat{k}h'} \rceil}(G) < \rho(r, k, m)$, then G has a k-factor.

(ii) For both odd r and odd k, and for $m = \lceil \frac{r}{k} \rceil$, if $r \leq kh^*$, or, if $r > kh^*$ and $\lambda_{\lceil \frac{2r}{r-kh^*} \rceil}(G) < \rho(r, k, m)$, then G has a k-factor.

(iii) For odd r and even k, and for $m = \lceil \frac{r}{r-k} \rceil$, if $r \leq (r-k)h^*$, or, if $r > (r-k)h^*$ and $\lambda_{\lceil \frac{2r}{r-(r-k)h^*} \rceil}(G) < \rho(r, k, m)$, then G has a k-factor.

Spectral Bound for odd $[1, b]$-factor

Theorem (Lu, Wu, and Yang 2010)

Let G be a connected r-regular graph of even order $n, r \geq 3$, and eigenvalues $r = \lambda_1 \geq \cdots \geq \lambda_n$. If one of the following conditions holds, G contains an odd $[1, b]$-factor:

(1) r is even, $\lceil \frac{r}{b} \rceil$ is even, and $\lambda_3 \leq r - \frac{\lceil \frac{r}{b} \rceil - 2}{r+1} + \frac{1}{(r+1)(r+2)}$,

(2) r is even, $\lceil \frac{r}{b} \rceil$ is odd, and $\lambda_3 \leq r - \frac{\lceil \frac{r}{b} \rceil - 1}{r+1} + \frac{1}{(r+1)(r+2)}$,

(3) r is odd, $\lceil \frac{r}{b} \rceil$ is even, and $\lambda_3 \leq r - \frac{\lceil \frac{r}{b} \rceil - 1}{r+1} + \frac{1}{(r+2)^2}$,

(4) r is odd, $\lceil \frac{r}{b} \rceil$ is odd, and $\lambda_3 \leq r - \frac{\lceil \frac{r}{b} \rceil - 2}{r+1} + \frac{1}{(r+2)^2}$.

A Sharp Spectral Bound for odd $[1, b]$-factor

Theorem (Kim, O, Park, and Park 2020)

Let G be a connected r-regular graph of even order $n, r \geq 3$, and eigenvalues $r = \lambda_1 \geq \cdots \geq \lambda_n$. If one of the following conditions holds, G contains an odd $[1, b]$-factor:

$$\lambda_3 < \begin{cases} \frac{r-2+\sqrt{(r+2)^2-4(\lceil \frac{r}{b} \rceil-2)}}{2} & \text{if both } r \text{ and } \lceil \frac{r}{b} \rceil \text{ are even} \\ \frac{r-2+\sqrt{(r+2)^2-4(\lceil \frac{r}{b} \rceil-1)}}{2} & \text{if } r \text{ is even and } \lceil \frac{r}{b} \rceil \text{ is odd} \\ \frac{r-3+\sqrt{(r+3)^2-4(\lceil \frac{r}{b} \rceil-2)}}{2} & \text{if both } r \text{ and } \lceil \frac{r}{b} \rceil \text{ are odd} \\ \frac{r-3+\sqrt{(r+3)^2-4(\lceil \frac{r}{b} \rceil-1)}}{2} & \text{if } r \text{ is odd and } \lceil \frac{r}{b} \rceil \text{ is even.} \end{cases}$$

Setting

Let $r_{ab} = \min\{r-a, b\}$, let

$$\eta = \begin{cases} \lceil \frac{r}{r_{ab}} \rceil - 1 \text{ if } r \text{ is even, } a \text{ and } b \text{ are odd, and } \lceil \frac{r}{r_{ab}} \rceil \text{ is odd,} \\ \lceil \frac{r}{r_{ab}} \rceil - 2 \text{ if } r \text{ is even, } a \text{ and } b \text{ are odd, and } \lceil \frac{r}{r_{ab}} \rceil \text{ is even,} \\ \lceil \frac{r}{b} \rceil - 1 \text{ if } r \text{ is odd, } a \text{ and } b \text{ are odd, and } \lceil \frac{r}{b} \rceil \text{ is even,} \\ \lceil \frac{r}{b} \rceil - 2 \text{ if } r \text{ is odd, } a \text{ and } b \text{ are odd, and } \lceil \frac{r}{b} \rceil \text{ is odd,} \\ \lceil \frac{r}{r-a} \rceil - 1 \text{ if } r \text{ is odd, } a \text{ and } b \text{ are odd, and } \lceil \frac{r}{r_{ab}} \rceil \text{ is even,} \\ \lceil \frac{r}{r-a} \rceil - 2 \text{ if } r \text{ is odd, } a \text{ and } b \text{ are odd, and } \lceil \frac{r}{r_{ab}} \rceil \text{ is odd,} \end{cases}$$

and let $\rho(r, a, b) = \begin{cases} \frac{r-2+\sqrt{(r+2)^2-4\eta}}{2} & \text{if } r \text{ is even,} \\ \frac{r-3+\sqrt{(r+3)^2-4\eta}}{2} & \text{if } r \text{ is odd.} \end{cases}$

A Sharp Spectral Bound for [a, b]-factor

Theorem (O, 2020 +)

Let r, a, b, h, h', and h^* be be positive integers such that $r \geq 3$, $a \leq b < r$, $h \leq r$, $h' \in \{h, h+1\}$ is an even number, and $h^* \in \{h, h+1\}$ is an odd number. Suppose that G is an h-edge-connected r-regular graph.

(i) For even r, odd a, b, and even $|V(G)|$, if $r \leq r_{ab}h'$, or if $r > r_{ab}h'$ and $\lambda_{\lceil \frac{2r}{r-r_{ab}h'} \rceil}(G) < \rho(r, a, b)$, then G has an odd [a, b]-factor.

(ii) For both odd r and odd a, b, if $r \leq bh^*$, or, if $r > bh^*$ and $\lambda_{\lceil \frac{2r}{r-bh^*} \rceil}(G) < \rho(r, a, b)$, then G has an odd [a, b]-factor.

(iii) For odd r and even a, b, if $r \leq (r-a)h^*$, or, if $r > (r-a)h^*$ and $\lambda_{\lceil \frac{2r}{r-(r-a)h^*} \rceil}(G) < \rho(r, a, b)$, then G has an even [a, b]-factor.

Tool

Lovasz's parity (g, f)-factor Theory

Let G be a graph and let g, f be two integer valued functions defined on $V(G)$ such that $0 \le g(v) \le f(v) \le d_G(v)$ and $g(v) \equiv f(v) \pmod 2$ for all $v \in V(G)$. Then G has a (g, f)-factor F such that $d_F(v) \equiv f(v) \pmod 2$ for all $v \in V(G)$ if and only if

$$\sum_{v \in T} (d(v) - g(v)) + \sum_{u \in S} f(u) - |[S, T]| - q(S, T) \ge 0$$

for all disjoint subsets S and T of $V(G)$, where $q(S, T)$ is the number of components Q of $G - (S \cup T)$ such that

$$|[V(Q), T]| + \sum_{v \in V(Q)} f(v) \equiv 1 \pmod 2.$$

Tool

Corollary

Let a and b be odd integers with $2 \leq a \leq b$. A graph G has an odd $[a, b]$-factor if and only if

$$\delta(S, T) := q(S, T) - b|S| + a|T| - \sum_{v \in T} d_{G-S}(v) \leq 0$$

for all disjoint subsets S and T of $V(G)$, where $q(S, T)$ is the number of components Q of $G - (S \cup T)$ such that $|[V(Q), T]| + b|V(Q)|$ is odd.

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Let a and b be even integers with $2 \leq a \leq b$. A graph G has an even $[a, b]$-factor if and only if

$$\delta(S, T) := q(S, T) - b|S| + a|T| - \sum_{v \in T} d_{G-S}(v) \leq 0$$

for all disjoint subsets S and T of $V(G)$, where $q(S, T)$ is the number of components Q of $G - (S \cup T)$ such that $|[V(Q), T]|$ is odd.

Question

Question. If G is an h-edge-connected r-regular graph and $a - b \equiv 1 \pmod 2$, then what are best upper bounds for a certain eigenvalue to guarantee the existence of an $[a, b]$-factor?

$a - b \equiv 1 \pmod 2$

Lovasz's (g, f)-factor Theory

Let G be a graph and let g, f be two integer valued functions defined on $V(G)$ such that $0 \le g(v) \le f(v) \le d_G(v)$ for all $v \in V(G)$. Then G has a (g, f)-factor F if and only if

$$\sum_{v \in T} (d(v) - g(v)) + \sum_{u \in S} f(u) - |[S, T]| - q(S, T) \ge 0$$

for all disjoint subsets S and T of $V(G)$, where $q(S, T)$ is the number of components Q of $G - (S \cup T)$ such that $g(v) = f(v)$ for all $v \in V(Q)$ and

$$|[V(Q), T]| + \sum_{v \in V(Q)} f(v) \equiv 1 \pmod 2.$$

Tool

Corollary

Let a and b be positive integers with $1 \le a < b$. A graph G has an $[a, b]$-factor if and only if

$$\delta(S, T) := -b|S| + a|T| - \sum_{v \in T} d_{G-S}(v) \le 0$$

for all disjoint subsets S and T of $V(G)$.

Thank you

Thank You :)

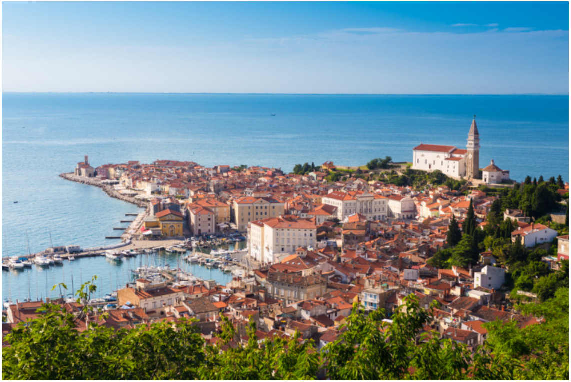

Figure: From Globe Guide