Stochastic completeness and uniqueness class for graphs

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Outline

1. Background
2. Weighted graphs
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Heat equation on $\mathbb{R}$

Consider the Cauchy problem for the heat equation:

$$
\begin{aligned}
\frac{\partial}{\partial t} u(x, t) + \Delta u(x, t) &= 0, \\
u(\cdot, 0) &\equiv 0;
\end{aligned}
$$

Here $u : \mathbb{R} \times [0, T] \to \mathbb{R}$, with $\Delta u(x, t) = -\frac{\partial^2}{\partial x^2} u(x, t)$. 
Tichonov solution

Natural solution: \( u \equiv 0 \);
Tichonov solution:

\[
\begin{align*}
  u(x, t) &= \sum_{k=0}^{\infty} \frac{g^k(t)}{(2k)!} x^{2k}, \\
  g(t) &= \begin{cases} 
    \exp(t^{-2}), & t > 0, \\
    0, & t \leq 0.
  \end{cases}
\end{align*}
\]

where

\[ |u(x, t)| \text{ can be bounded at best by } \exp\left( C(\varepsilon) |x|^{2+\varepsilon} \right). \]
Täcklind’s uniqueness class

Täcklind proved that if $|u(x, t)| \leq h(|x|)$ for $|x|$ large, where

$$
\int_{-\infty}^{\infty} \frac{r}{\ln h(r)} \, dr = +\infty,
$$

then $u \equiv 0$. The solution to the Cauchy problem (♠) is unique in such a class of functions. In particular, bounded functions form a uniqueness class.
Stochastic completeness

The Laplacian $\Delta$ generates a semigroup of operators

$$P_t = \exp(-t\Delta), \ t \geq 0.$$ 

It is closely related to the Brownian motion $(B_t)_{t \geq 0}$ on $\mathbb{R}$:

$$P_tf(x) = \mathbb{E}_x(f(B_t)).$$

Bounded solutions form a uniqueness class $\iff$ stochastic completeness, that is,

$$P_t1 = 1.$$ 

(Note that $1 - P_t1$ is a bounded solution to the Cauchy problem.)
Heat equation on manifolds

Let \((M, g)\) be a complete Riemannian manifold with the Laplace-Beltrami operator \(\Delta \geq 0\). Consider the Cauchy problem for the heat equation:

\[
\begin{cases}
\frac{\partial}{\partial t} u(x, t) + \Delta u(x, t) = 0, \\
u(\cdot, 0) \equiv 0.
\end{cases}
\]
Grigor’yan’s uniqueness class

Theorem (Grigor’yan)

If \( u : M \times [0, T] \to \mathbb{R} \) solves (♠) and satisfies

\[
\int_0^T \int_{B(\bar{x}, r)} u^2(x, t) \, d\text{vol}(x) \, dt \leq h(r)
\]

for \( r \) large, where

\[
\int_0^\infty \frac{r}{\ln h(r)} \, dr = +\infty,
\]

then \( u \equiv 0 \).
Proof strategy

A localized version of monotonicity formula:

\[
\frac{d}{dt} \int_M u^2(x, t) \exp \xi(x, t) \, d\text{vol}(x) \leq 0,
\]

where \( \xi \) satisfies

\[
\frac{\partial}{\partial t} \xi(x, t) + \frac{1}{2} |\nabla \xi(x, t)|^2 \leq 0.
\]

For example: \( \xi(x, t) = -\frac{d(\bar{x}, x)^2}{2t} \).
Stochastic completeness

The Laplacian $\Delta$ generates a semigroup of operators

$$ P_t = \exp(-t\Delta), \quad t \geq 0. $$

It is closely related to the Brownian motion $(B_t)_{t \geq 0}$ on $(M, g)$:

$$ P_t f(x) = \mathbb{E}_x(f(B_t)). $$

Bounded solutions form a uniqueness class $\iff$ stochastic completeness, that is,

$$ P_t 1 = 1. $$
Volume growth criteria for stochastic completeness

The uniqueness class theorem, when applied to bounded solutions, implies a sharp volume growth type criterion for stochastic completeness.

**Theorem (Grigor’yan)**

Suppose

$$\int_{\infty}^{\infty} \frac{rdr}{\ln(\text{vol}(B_d(\bar{x}, r)))} = \infty,$$  \hspace{1cm} (†)

then the Brownian motion on \((M, g)\) is stochastically complete.
Heat equation on $\mathbb{Z}$

What happens for graphs?

$\begin{cases}
\frac{\partial}{\partial t} u(x, t) + \Delta u(x, t) = 0,
\end{cases}$

$u(\cdot, 0) \equiv 0.$

Here $u : \mathbb{Z} \times [0, T] \rightarrow \mathbb{R}$, with

$\Delta u(n, t) = 2u(n, t) - (u(n - 1, t) + u(n + 1, t)).$
Tichonov type solution

Natural solution: \( u \equiv 0 \);

Tichonov type solution:

\[
\begin{align*}
  u(n, t) &= \begin{cases} 
    g(t), & n = 0; \\
    \sum_{k=0}^{\infty} \frac{g^k(t)}{(2k)!} (n + k) \cdots (n + 1) n \cdots (n - k + 1), & n \geq 1; \\
    u(-n - 1, t), & n \leq -1.
  \end{cases}
\end{align*}
\]

where

\[
  g(t) = \begin{cases} 
    \exp(t^{-2}), & t > 0, \\
    0, & t \leq 0.
  \end{cases}
\]
Growth

Note that for \( n \geq 1 \),

\[
\sum_{k=0}^{\infty} \frac{g^k(t)}{(2k)!} (n + k) \cdots (n + 1)n \cdots (n - k + 1) = \sum_{k=0}^{n} \cdots .
\]

In contrary to the smooth case, for large \( |n| \),

\[
|u(n, t)| \leq \exp(C |n| \ln |n|).
\]
Questions

• What about the uniqueness class for general weighted graphs? We cannot expect growth conditions as large as the smooth case.

• What is the sharp volume growth type criterion for stochastic completeness of weighted graphs?
Weighted graphs

Let $V$ be a discrete countable set with weights:

- $\mu : V \to (0, \infty)$, as a measure on $V$;
- $w : V \times V \to [0, \infty)$
  - a. $w(x, y) = w(y, x)$;
  - b. $w(x, x) = 0$;
  - c. $\sum_{y \in V} w(x, y) < +\infty$.

Denote $x \sim y$ when $w(x, y) > 0$. We assume connectedness.

The formal Laplacian:

$$(\Delta f)(x) = \frac{1}{\mu(x)} \sum_{y \in V} w(x, y)(f(x) - f(y)).$$
The heat semigroup

The Laplacian $\Delta$ generates the heat semigroup

$$P_t = \exp(-t\Delta), \quad t \geq 0,$$

which corresponds to a minimal continuous time Markov chain on $V$.

Bounded solutions form a uniqueness class for the Cauchy problem ($\spadesuit$) of the heat equation $\iff$ stochastic completeness, that is,

$$P_t1 = 1.$$
\( \mathbb{Z} \) with weights

\[ V = \mathbb{Z}, \text{ with } n \sim n + 1, \text{ as a graph.} \]

- \( \mu(n) \equiv 1, \ w(n, n + 1) \equiv 1; \)
- \( \mu(n) \equiv 1, \ w(0, -1) = 1, \)
  \( w(n - 1, n) = w(-n, -n - 1) = n \) for \( n \geq 1. \)
Intrinsic metrics

**Definition**

A metric $d$ on $(V, w, \mu)$ is called an intrinsic metric if

$$\forall x \in V, \quad \frac{1}{\mu(x)} \sum_{y \in V} w(x, y)d(x, y)^2 \leq 1. \quad (\Diamond)$$

**Remark**

An intrinsic metric is sensible to the weights $\mu, w$. Condition $(\Diamond)$ is a discrete analogue of $|\nabla d(\bar{x}, \cdot)| \leq 1$. For simplicity, we also assume bounded jump size: $d(x, y) \leq \sigma_0$ whenever $x \sim y$. 
Examples of intrinsic metrics

\[ V = \mathbb{Z}, \text{ with } n \sim n + 1, \text{ as a graph.} \]

- \( \mu(n) \equiv 1, \ w(n, n + 1) \equiv 1; \text{ let } d(n, n + 1) \equiv \frac{\sqrt{2}}{2} \) which is naturally extended to a shortest path metric.

- \( \mu(n) \equiv 1, \ w(0, -1) = 1, \ w(n - 1, n) = w(-n, -n - 1) = n \text{ for } n \geq 1; \text{ let } \)

\[
d(n - 1, n) = \sqrt{\frac{1}{2 \vee (2 |n| + 1)}},
\]

which is naturally extended to a shortest path metric.
Uniqueness class

**Theorem (H.)**

*Under some mild conditions, for some constant $c > 0$, if $u : V \times [0, T] \to \mathbb{R}$ solves the Cauchy problem $\mathcal{L}$ and satisfies*

$$
\int_0^T \int_{B(\bar{x}, r)} u^2(x, t) \, d\mu(x) \, dt \leq \exp(c\sigma_0 r \ln r)
$$

*for $r$ large, then $u \equiv 0$.*

**Remark**

*As a consequence, if $\mu(B(\bar{x}, r)) \leq \exp(c\sigma_0 r \ln r)$ for $r$ large, then the corresponding Markov chain is stochastically complete.*
Difficulties

Lack of chain rule: unlike

$$|\nabla \exp \xi(x)| \leq \exp \xi(x) |\nabla \xi(x)|,$$

we have at best

$$\frac{1}{\mu(x)} \sum_{y \in V} w(x, y) (\exp \xi(x) - \exp \xi(y))^2$$

$$\leq \exp 2(\xi(x) \lor \xi(y)) \frac{1}{\mu(x)} \sum_{y \in V} w(x, y) (\xi(x) - \xi(y))^2.$$
Stochastic completeness

**Theorem (Folz)**

*Under some technical conditions, if*

\[
\int_{0}^{\infty} \frac{r \, dr}{\ln \mu(B(\bar{x}, r))} = \infty,
\]

*then \((V, w, \mu)\) is stochastically complete.*

**Remark**

*Folz works by relating the Markov chain to a diffusion. Stochastic completeness is about “very large” time property, and is much more stable than the uniqueness class is (which involves short time information as well).*
Goals of the present work

- to recover Grigor’yan’s uniqueness class for a certain special class of weighted graphs;
- to apply stability arguments to obtain a generalized version of Folz’s volume growth criterion.
GL (globally local) condition

Let

$$s_r := \sup \{ d(x, y) \mid x, y \in X \text{ with } x \sim y \text{ and } d(x, \bar{x}) \land d(y, \bar{x}) \geq r \}.$$  

Definition

A weighted graph \((V, w, \mu)\) with an intrinsic metric \(d\) is called globally local with respect to an increasing function \(f : (0, \infty) \to (0, \infty)\) if there is a constant \(A > 1\) such that

$$\limsup_{r \to \infty} \frac{s_r f(Ar)}{r} < \infty. \quad \text{(GL)}$$
Uniqueness class under the GL condition

Theorem (H., Keller, Schmidt)

Let a weighted graph \((V, w, \mu)\) with an intrinsic metric \(d\) be globally local with respect to an increasing function \(f : (0, \infty) \rightarrow (0, \infty)\) with \(\int_{\infty}^{\infty} \frac{r}{f(r)} \, dr = +\infty\). Assume that balls in \(d\) are finite. If \(u : V \times [0, T] \rightarrow \mathbb{R}\) solves the Cauchy problem \((\spadesuit)\) and satisfies

\[
\int_{0}^{T} \int_{B(\bar{x}, r)} u^2(x, t) \, d\mu(x) \, dt \leq \exp f(r)
\]

for \(r\) large, then \(u \equiv 0\).
Stochastic completeness

**Theorem (H., Keller, Schmidt)**

Let \((V, w, \mu)\) be a weighted graph with an intrinsic metric \(d\) such that balls in \(d\) are finite. If

\[
\int_{\infty} r \, dr \left( \frac{r \, dr}{\ln \mu(B(\bar{x}, r))} \right) = \infty,
\]

then \((V, w, \mu)\) is stochastically complete.
Stability and modifications of weighted graphs

Main ingredients:

• a “piecing out” argument to deal with unbounded jump size;

• adding new vertices to the original weighted graph to split big jumps into smaller steps (a globally local one);

• a potential theoretic argument (the weak Omori-Yau maximum principle) for stability of stochastic completeness under modifications.
A sharpness example

\[ V = \mathbb{Z}, \text{ with } n \sim n + 1, \text{ as a graph. Given weights } \mu(n) \equiv 1, \]
\[ w(0, -1) = 1, \ w(n - 1, n) = w(-n, -n - 1) = n \text{ for } n \geq 1. \text{ Let } \]
\[ d(n - 1, n) = \sqrt{\frac{1}{2 \vee (2|n| + 1)}}, \]

which is naturally extended to a shortest path metric.
A sharpness example

We have $d(0, n) \simeq \sqrt{|n|}$, and $s_r \simeq \frac{1}{r}$ for $r$ large.

A Tichonov type solution:

$$u(n, t) = \begin{cases} 
  g(t), & n = 0; \\
  \sum_{k=0}^{\infty} \binom{n}{k} \frac{g^k(t)}{k!}, & n \geq 1; \\
  u(-n - 1, t), & n \leq -1.
\end{cases}$$
A sharpness example

Bound:

\[
\int_0^T \int_{B(\bar{x},r)} u^2(x, t) \, d\mu(x) \, dt \leq \exp(Cr^2 \ln r)
\]

for \( r \) large.

Note

\[
\frac{s_r f(Ar)}{r} \sim \ln r.
\]

This example fails to be globally local with respect to \( f(r) = Cr^2 \ln r \) (roughly by a factor of \( \ln r \)), and a Tichonov type solution is present.
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Thank you very much!