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- Introduction
- The Classical Risk Model
- The Diffusion Approximation
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Hanspeter Schmidli
Optimal Drawdowns in Insurance
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Definition of Drawdown

For a surplus process $X_t$ denote by

$$\bar{X}_t = \max\{\bar{x}, \sup_{s \leq t} X_s\}$$

the running maximum. The drawdown

$$D_t = \bar{X}_t - X_t$$

is the deviation from the running maximum. We allow a past maximum $\bar{x}$.
Interpretation

- Large drawdowns are a reputational risk
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Investors compare with current maximum (risk for the manager)
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- Goal is to keep surplus near maximum (stabilisation) which simplifies planning future strategies
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We try to keep the drawdown below some level $d$
Drawdowns

Introduction

Interpretation

- Large drawdowns are a reputational risk
- Investors compare with current maximum (risk for the manager)
- Goal is to keep surplus near maximum (stabilisation) which simplifies planning future strategies
- We try to keep the drawdown below some level $d$
- Drawdown below the critical level only for a short time
Reinsurance

The insurer buys proportional reinsurance with retention level $b_t \in [0, 1]$ at time $t$. That is, the insurer pays $b_t Y$, the reinsurer $(1 - b_t) Y$ of a claim of size $Y$. The reinsurer uses an expected value principle with safety loading $\theta$. We assume that reinsurance is more expensive than first insurance in order that the problem below is not trivial. The insurer chooses continuously a reinsurance strategy $\{b_t\}$. 
The Optimisation Problem

The value of a reinsurance strategy \( b \) is

\[
V^b(x) = \mathbb{E}\left[ \int_0^\infty e^{-\delta t} \mathbb{I}_{D_t^b > d} \, dt \right].
\]

We are interested in the optimal value

\[
V(x) = \inf_b V^b(x)
\]

and, if it exist, the optimal strategy \( b^* \).
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The Cramér–Lundberg Model

Let

\[ X_t = \bar{x} - x + ct - \sum_{k=1}^{N_t} Y_k, \]

where \( N \) is a Poisson process with rate \( \lambda \) and iid claim \( \{Y_k\} \) with expected value \( \mu \). We write \( c = (1 + \eta)\lambda \mu \) for some \( \eta > 0 \).
The Cramér–Lundberg Model

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After reinsurance,

\[ X_t^b = \bar{x} - x + \int_0^t c(b_s) \, ds - \sum_{k=1}^{N_t} b_{T_k} Y_k , \]

where \( c(b) = c - (1 - b)(1 + \theta)\lambda\mu = (b\theta - (\theta - \eta))\lambda\mu \).
The Classical Risk Model

The Drawdown Process

We get the drawdown process

\[ D_t^b = x + \sum_{k=1}^{N_t} b_{T_k} - Y_k - \int_0^t c(b_s) \, ds + (\bar{X}_t^b - \bar{x}) . \]
We get the drawdown process

\[ D_t^b = x + \sum_{k=1}^{N_t} b_{T_k} - Y_k - \int_0^t c(b_s) \, ds + (\bar{X}_t^b - \bar{x}) . \]

That is

- jumps upwards, (downwards) deterministic paths
We get the drawdown process

\[ D^b_t = x + \sum_{k=1}^{N_t} b_{T_k} - Y_k - \int_0^t c(b_s) \, ds + (\bar{X}^b_t - \bar{x}). \]

That is

- jumps upwards, (downwards) deterministic paths
- reflection in zero
The Drawdown Process

We get the drawdown process

\[ D_t^b = x + \sum_{k=1}^{N_t} b_{T_k} - Y_k - \int_0^t c(b_s) \, ds + (\bar{X}_t^b - \bar{x}) . \]

That is

- jumps upwards, (downwards) deterministic paths
- reflection in zero

- we now restrict to \( b_t \in [b^0, 1] \) with \( b^0 = (1 - \eta/\theta) \), such that \( c(b_t) \geq 0 \).
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With simplified notation, the diffusion approximation to the classical model is \( X_t = \bar{x} - x + \eta t + \sigma W_t \) for some Brownian motion \( W_t \). After reinsurance

\[
X_t^b = \bar{x} - x + \int_0^t \{ b_s \theta - (\theta - \eta) \} \, ds + \sigma \int_0^t b_s \, dW_s.
\]
With simplified notation, the diffusion approximation to the classical model is $X_t = \bar{x} - x + \eta t + \sigma W_t$ for some Brownian motion $W$. After reinsurance

$$X^b_t = \bar{x} - x + \int_0^t \{ b_s \theta - (\theta - \eta) \} \, ds + \sigma \int_0^t b_s \, dW_s .$$

The drawdown process becomes

$$D^b_t = x - \int_0^t \{ b_s \theta - (\theta - \eta) \} \, ds - \sigma \int_0^t b_s \, dW_s + (\bar{X}^b_t - \bar{x}) .$$
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Lipschitz Continuity

Lemma

The function \( V \) is increasing with \( 0 \leq V(x) \leq \delta^{-1} \) for all \( x \in [0, \infty) \), fulfils \( \lim_{x \to \infty} V(x) = \delta^{-1} \) and is Lipschitz continuous with

\[
|V(x) - V(y)| \leq \frac{\lambda + \delta}{\delta c(1)} |x - y|.
\]

In particular, \( V \) is absolutely continuous and differentiable almost everywhere.
Proof.

For $0 \leq y < x$ choose a strategy $\tilde{b}$ with $V^{\tilde{b}}(y) < V(y) + \varepsilon$. For initial capital $x$ define $h = (x - y)/c(1)$, $b_t = \tilde{b}_{t-h}$ if $T_1 \wedge t \geq h$ and $b_t = 1$, otherwise. Then

$$V(x) - V(y) - \varepsilon \leq V^b(x) - V^{\tilde{b}}(y)$$

$$\leq \int_0^h e^{-\delta t} \, dt - (1 - e^{-(\lambda+\delta)h}) V^{\tilde{b}}(y) + (1 - e^{-\lambda h}) \delta^{-1}$$

$$\leq (\lambda + \delta) h/\delta = \frac{\lambda + \delta}{\delta c(1)} (x - y),$$
Proof.

For $0 \leq y < x$ choose a strategy $\tilde{b}$ with $V^{\tilde{b}}(y) < V(y) + \varepsilon$. For initial capital $x$ define $h = (x - y) / c(1)$, $b_t = \tilde{b}_{t-h}$ if $T_1 \land t \geq h$ and $b_t = 1$, otherwise. Then

$$V(x) - V(y) - \varepsilon \leq V^b(x) - V^{\tilde{b}}(y)$$

$$\leq \int_0^h e^{-\delta t} \, dt - (1 - e^{-(\lambda+\delta)h})V^{\tilde{b}}(y) + (1 - e^{-\lambda h})\delta^{-1}$$

$$\leq (\lambda + \delta)h / \delta = \frac{\lambda + \delta}{\delta c(1)}(x - y),$$

The other statements are clear.
Splitting of the Problem

Let

\[ \vartheta_d = \inf \{ t \geq 0 : D_t \leq d \} , \quad \vartheta^d = \inf \{ t \geq 0 : D_t > d \} \]

be the first entrance times. Then by considering the process until the stopping time

\[
V(x) = \mathbb{E}[e^{-\delta \vartheta_d} V(D_{\vartheta_d})] , \quad x \leq d , \\
V(x) = \mathbb{E}[\delta^{-1}(1 - e^{-\delta \vartheta_d}) + e^{-\delta \vartheta_d} V(d)] , \quad x > d .
\]

We can solve the two problems separately.
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The Solution

Starting in the Critical Area

Problem: Maximise $\mathbb{E}^x[e^{-\delta y_d}]$. 
Problem: Maximise $\mathbb{E}^x[e^{-\delta \vartheta_d}]$.

For $x > d$, reaching $d$ one has to pass $y \in (d, x)$. Conclusion: $\mathbb{E}^x[e^{-\delta \vartheta_d}]$ is an exponential function.
Starting in the Critical Area

Problem: Maxmise $\mathbb{E}^{x}[e^{-\delta \vartheta_d}]$.

For $x > d$, reaching $d$ one has to pass $y \in (d, x)$. Conclusion: $\mathbb{E}^{x}[e^{-\delta \vartheta_d}]$ is an exponential function.

For any subinterval of a fixed length, the same quantity has to be maximised. Conclusion: the optimal strategy is constant.
Starting in the Critical Area

Problem: Maximise $\mathbb{E}^x[e^{-\delta y_d}]$.

For $x > d$, reaching $d$ one has to pass $y \in (d, x)$. Conclusion: $\mathbb{E}^x[e^{-\delta y_d}]$ is an exponential function.

For any subinterval of a fixed length, the same quantity has to be maximised. Conclusion: the optimal strategy is constant.

$$V(x) = \delta^{-1} - (\delta^{-1} - V(d))e^{-\gamma(x-d)}$$

where $\gamma$ is the positive solution to $c(1)\gamma - \lambda\mathbb{E}[1 - e^{-\gamma Y}] = \delta$. 
Starting in the Non-Critical Area

Problem: Minimise \( \mathbb{E}[e^{-\delta d} V(D_{\vartheta d})] \) with \( V(d) \) unknown.
Starting in the Non-Critical Area

Problem: Minimise \( \mathbb{E}[e^{-\delta \vartheta} V(D_{\vartheta d})] \) with \( V(d) \) unknown.

Replace \( V(d) \) by \( C \in (0, \delta^{-1}) \), \( V_C(x) = \inf_b \mathbb{E}[e^{-\delta \vartheta} V_C(D_{\vartheta d})] \).

**Lemma**

There exists \( C_0 \in (0, \delta^{-1}) \) such that \( V_C(d) \gtrless C \) iff \( C \gtrless C_0 \).

It turns out that \( C_0 = V(d) \).
The HJB Equation

**Theorem**

$V_C(x)$ solves for $x \leq d$ the HJB equation

$$\inf_{b \in [b^0, 1]} \lambda \int_0^\infty V_C(x + by) \, dG(y) - c(b) V'_C(x) - (\lambda + \delta) V_C(x) = 0.$$ 

Let $b_C(x)$ be a measurable version of the maximiser. Then the strategy $b_C(D^C_t)$ is optimal.
The HJB Equation II

Theorem

$V(x)$ is the unique bounded continuous solution to the HJB equation

$$\inf_{b \in [b^0, 1]} \lambda \int_0^\infty V(x+by) \, dG(y) - c(b)V'(x) - (\lambda + \delta)V(x) = -\mathbb{I}_{x > d}.$$  

Let $b(x)$ be a measurable version of the maximiser. Then the strategy $b(D_t^*)$ is optimal.
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Exponentially Distributed Claims

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Exponentially Distributed Claims: No Reinsurance
Exponentially Distributed Claims: Linear Reinsurance
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Splitting of the Problem

As for the classical model

\[
V(x) = \mathbb{E}[e^{-\delta \vartheta} V(d)] , \quad x \leq d ,
\]

\[
V(x) = \mathbb{E}[\delta^{-1}(1 - e^{-\delta \vartheta}) + e^{-\delta \vartheta} V(d)] , \quad x > d .
\]

In the critical area \( x > d \) \( b = 1 \) and thus

\[
V(x) = \delta^{-1}\{1 - (1 - \delta V(d))e^{-\kappa(x-d)}\}
\]

for \( \kappa > 0 \) solving \( \frac{1}{2} \sigma^2 \kappa^2 + \eta \kappa = \delta \).
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The HJB Equation

**Theorem**

$V(x)$ is the unique bounded continuously differentiable solution to

$$(\theta - \eta)V'(x) - \delta V(x) + \inf_{b \in [0,1]} \left\{ \frac{1}{2} b^2 \sigma^2 V''(x) - \theta b V'(x) \right\} = -\mathbb{I}_{x > d}.$$ 

**Proof.**

Explicit solution to the HJB and verification theorem.
A non-trivial solution must be strictly convex. If \( b \neq 1 \),

\[
\frac{\theta^2 V'(x)^2}{2\sigma^2 V''(x)} + \delta V(x) = (\theta - \eta) V'(x) .
\]

The function \( x \mapsto -\ln V'(x) \) is strictly decreasing with inverse function \( Y \). Thus \( V'(Y(z)) = e^{-z} \). Plugging this into the equation and differentiation leads to differential equation and an explicit solution. There is \( x_0 \in (0, \infty] \) such that

\[
b(x) = \frac{\theta V'(x)}{\sigma^2 V''(x)} \leq 1 , \quad x \in [0, x_0] .
\]

Compound \( V(x) \) on \([0, x_0 \wedge d]\) with the solution with \( b(x) = 1 \) to a smooth solution.
The Behaviour at Zero

**Theorem**

The strategy $b(D_t^*)$ is optimal. Under the optimal strategy $\bar{X}_t^*$ is constant.
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Value Function and $b(x)$

$\nu(\theta, x)$

$b(\theta, x)$
No Reinsurance
Optimal Reinsurance
References


Thank you for your attention