Strictly additive 2-designs

Anamari Nakic

University of Zagreb

Joint work with Marco Buratti

Portorož, June 2021

This work has been supported by HRZZ grant no. 9752 and no. 6732

Definition (2-Design)

A 2-(v, k, λ) design is a pair $(\mathcal{P}, \mathcal{B})$ such that

- $\mathcal{P}$ is a set of v points;
- $\mathcal{B}$ is a collection of k-subsets of $\mathcal{P}$ (called blocks);
- each 2-subset of $\mathcal{P}$ is contained in λ blocks.

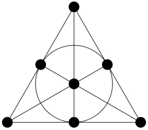

Figure: The Fano plane. 2-$(7, 3, 1)$ design.

- A 2-design is symmetric if $|\mathcal{P}| = |\mathcal{B}|$.
- A Steiner system is a design with $\lambda = 1$.

Definition (Cageggi, Falcone, Pavone, 2017)

A design $(\mathcal{P}, \mathcal{B})$ is additive under an abelian group G if

- $\mathcal{P} \subseteq G$ and
- $\sum_{x \in B} x = 0, \forall B \in \mathcal{B}$.

Examples:

Parameters	Group	Description
$(p^{mn}, p^m, 1)$	$\mathbb{Z}_p^{mn}$	points and lines of $AG(n, p^m)$
$(2^n - 1, 3, 1)$	$\mathbb{Z}_2^n$	points and lines of $PG(n-1, 2)$

Definition (Design)

A 2-(v, k, λ) design is a pair $(\mathcal{P}, \mathcal{B})$ such that

- $\mathcal{P}$ is a set of v points;
- $\mathcal{B}$ is a collection of k-subsets of $\mathcal{P}$ (called blocks);
- each 2-subset of $\mathcal{P}$ is contained in λ blocks.

Definition (Cameron, 1974. Delsarte, 1976.)

A 2-(v, k, λ) design over $\mathbb{F}_q$ is a pair $(\mathcal{P}, \mathcal{B})$ such that

- $\mathcal{P}$ is the set of points of $\mathrm{PG}(v-1, q)$
- $\mathcal{B}$ is a collection of $(k-1)$-dimensional subspaces $\mathrm{PG}(v-1, q)$ (blocks)
- each line is contained in λ blocks.

- Greferath, Pavcevic, Silberstein, Vazquez-Castro. *Network Coding and Subspace Designs*, Springer, 2018

Definition (Cameron, 1974. Delsarte, 1976.)

A 2-(v, k, λ) design over $\mathbb{F}_q$ is a pair $(\mathcal{P}, \mathcal{B})$ such that

- $\mathcal{P}$ is the set of points of $\mathrm{PG}(v-1, q)$
- $\mathcal{B}$ is a collection of $(k-1)$-dimensional subspaces $\mathrm{PG}(v-1, q)$ (blocks)
- each line is contained in λ blocks.

Properties:

- 2-(v, k, λ) design over $\mathbb{F}_q$ is a classical 2-$(\frac{q^v-1}{q-1}, \frac{q^k-1}{q-1}, \lambda)$ design
- 2-(v, k, λ) design over $\mathbb{F}_2$ is additive in $\mathbb{Z}_2^v$

Parameters	Description	Reference
2-$(2^v-1, 7, 7)$, v odd	2-$(v, 3, 7)$ design over $\mathbb{F}_2$ for all v odd	Thomas, 1987 + Buratti, A.N., 2019
2-$(8191, 7, 1)$	2-$(13, 3, 1)$ design over $\mathbb{F}_2$	Braun, Etzion, Ostergaard, Vardy, Wassermann, 2017

Definition

$(\mathcal{P}, \mathcal{B})$ is additive under an abelian group G if $\mathcal{P} \subseteq G$ and $\sum_{x \in B} x = 0, \forall B \in \mathcal{B}$.

- strongly additive if $\mathcal{B} = \{B \in \binom{\mathcal{P}}{k} \mid \sum_{x \in B} x = 0\}$
- strictly additive if $\mathcal{P} = G$
- almost strictly additive if $\mathcal{P} = G \setminus \{0\}$

[Cageggi, Falcone, Pavone, 2017]

Parameters	Group	Strongly	Strictly	Almost str.	Description
$(2^n - 1, 3, 1)$	$\mathbb{Z}_2^n$	✓		✓	points and lines of $PG(n-1, 2)$
$(p^{mn}, p^m, 1)$	$\mathbb{Z}_p^{mn}$		✓		points and lines of $AG(n, p^m)$
$(p^2, p, 1)$	$\mathbb{Z}_p^{\frac{p(p-1)}{2}}$	✓			points and lines of $AG(2, p)$
(v, k, λ)	$\mathbb{Z}_k \times \mathbb{Z}_{k-\lambda}^{\frac{v-1}{2}}$	✓			symmetric design, $k - \lambda \not\mid k$, prime
(v, k, λ)	G	✓			symmetric design

Definition

$(\mathcal{P}, \mathcal{B})$ is additive under an abelian group G if $\mathcal{P} \subseteq G$ and $\sum_{x \in B} x = 0, \forall B \in \mathcal{B}$.

- strongly additive if $\mathcal{B} = \{B \in \binom{\mathcal{P}}{k} \mid \sum_{x \in B} x = 0\}$
- strictly additive if $\mathcal{P} = G$
- almost strictly additive if $\mathcal{P} = G \setminus \{0\}$

[Buratti, A.N., 202?]

Parameters	Group	Strongly	Strictly	Almost str.	Description
$(2^v - 1, 2^k - 1, \lambda)$	$\mathbb{Z}_2^v$			✓	(v, k, λ) design over $\mathbb{F}_2$, in $PG(v-1, 2)$
$(8191, 7, 1)$	$\mathbb{Z}_2^{13}$			✓	$(13, 3, 1)$ design over $\mathbb{F}_2$, in $PG(12, 2)$

[A.N., Examples and Counterexamples, 2021]

Parameters	Group	Strongly	Strictly	Almost str.	Description
$(81, 6, 2)$	$\mathbb{Z}_3^4$		✓		each block is a union of two parallel lines of AG$(4, 3)$

Properties:

- it is simple
- the only known 2-$(81, 6, 2)$ has repeated blocks (Hanani, 1975)
- 432 blocks are obtained from 16 orbits of $\mathbb{Z}_3^4$ of size 27 (representatives bellow)

$$\{(0,0,0,0),(0,0,0,1),(0,0,0,2),(0,1,0,0),(0,1,0,1),(0,1,0,2)\}$$
$$\{(0,0,0,0),(0,0,1,1),(0,0,2,2),(2,1,0,0),(2,1,1,1),(2,1,2,2)\}$$
$$\{(0,0,0,0),(0,1,1,1),(0,2,2,2),(0,0,1,0),(0,1,2,1),(0,2,0,2)\}$$
$$\{(0,0,0,0),(0,1,2,0),(0,2,1,0),(2,0,2,1),(2,1,1,1),(2,2,0,1)\}$$
$$\{(0,0,0,0),(1,0,0,0),(2,0,0,0),(0,2,2,1),(1,2,2,1),(2,2,2,1)\}$$
$$\{(0,0,0,0),(1,0,1,0),(2,0,2,0),(0,1,0,0),(1,1,1,0),(2,1,2,0)\}$$
$$\{(0,0,0,0),(1,0,1,1),(2,0,2,2),(0,0,2,0),(1,0,0,1),(2,0,1,2)\}$$
$$\{(0,0,0,0),(1,0,2,0),(2,0,1,0),(0,2,1,1),(1,2,0,1),(2,2,2,1)\}$$
$$\{(0,0,0,0),(1,0,2,2),(2,0,1,1),(0,1,2,1),(1,1,1,0),(2,1,0,2)\}$$
$$\{(0,0,0,0),(1,1,0,0),(2,2,0,0),(0,2,0,1),(1,0,0,1),(2,1,0,1)\}$$
$$\{(0,0,0,0),(1,1,0,1),(2,2,0,2),(0,2,2,0),(1,0,2,1),(2,1,2,2)\}$$
$$\{(0,0,0,0),(1,1,2,0),(2,2,1,0),(0,0,2,1),(1,1,1,1),(2,2,0,1)\}$$
$$\{(0,0,0,0),(1,1,2,1),(2,2,1,2),(0,2,1,1),(1,0,0,2),(2,1,2,0)\}$$
$$\{(0,0,0,0),(1,1,2,2),(2,2,1,1),(0,2,2,0),(1,0,1,2),(2,1,0,1)\}$$
$$\{(0,0,0,0),(1,2,1,2),(2,1,2,1),(0,0,2,1),(1,2,0,0),(2,1,1,2)\}$$
$$\{(0,0,0,0),(1,2,2,0),(2,1,1,0),(0,2,2,1),(1,1,1,1),(2,0,0,1)\}$$

Definition

A 2-(q^n, kq, λ) design $(\mathcal{P}, \mathcal{B})$ is k-parallel if

- $\mathcal{P}$ is the set of points of AG(n, q),
- each block $B \in \mathcal{B}$ is union of k parallel lines of AG(n, q).

Parameters	Group	Strongly	Strictly	Al. str.	Description
k-parallel	$\mathbb{Z}_q^n$		✓		each block is a union of k parallel lines of AG(n, q)

Definition (Difference Set)

- G additive group
- k-subset D of G is a (G, k, λ) difference set if each non-zero element of G is covered λ times by the list of differences of D:

$$\Delta D = \{x - y \,:\, x \neq y, x, y \in D\} = \lambda\,(G \setminus \{0\})$$

Definition (Difference Family)

- G additive group
- A collection of k-subsets $\mathcal{F} = \{D_1, \ldots, D_t\}$ of G is a (G, k, λ) difference family if each non-zero element of G is covered λ times by the list of differences of the blocks:

$$\Delta \mathcal{F} = \uplus \Delta D_i = \lambda\,(G \setminus \{0\})$$

Theorem (Buratti, A.N., 202?)

If there exists a (q, k, λ) *difference family in* $\mathbb{F}_q$ *then there exists a strictly additive 2-*(q^n, kq, μ) *design under* $(\mathbb{F}_q^n, +)$ *with* $\mu = \frac{\lambda(kq-1)}{k-1}$*, for every* $n \geq 2$*.*

Proof.

Difference family $\Rightarrow$ k-parallel design $\Rightarrow$ strictly additive design □

Another example:

Parameters	Group	Strongly	Strictly	Al. str.	Description
$(49, 21, 10)$	$\mathbb{Z}_7^2$		✓		$(7, 3, 1)$ difference set

Properties:

- it is simple
- each block is a union of 3 parallel lines of $AG(2, 7)$
- not isomorphic to the design of Abel, 1996

Corollary [Buratti, A.N., 202?]

Parameters	Group	Strictly	Description	Reference
$(q^n, 2q, 2q-1)$	$\mathbb{Z}_q^n$	✓	$(q,2,1)$ DF, q odd	patterned starter
$(q^n, 3q, \frac{3q-1}{2})$	$\mathbb{Z}_q^n$	✓	$(q,3,1)$ DF, $q \equiv 1 \pmod 6$	Peltesohn, 1938
$(q^n, 4q, \frac{4q-1}{3})$	$\mathbb{Z}_q^n$	✓	$(q,4,1)$ DF, $q \equiv 1 \pmod{12}$	Chen, Zhu, 1999
$(q^n, 5q, \frac{5q-1}{4})$	$\mathbb{Z}_q^n$	✓	$(q,5,1)$ DF, $q \equiv 1 \pmod{20}$	Chen, Zhu, 1999
$(q^n, 6q, \frac{6q-1}{5})$	$\mathbb{Z}_q^n$	✓	$(q,6,1)$ DF, $q \equiv 1 \pmod{30}$ except possibly $q=61$	Chen, Zhu, 1998
$(q^n, \frac{q(q-1)}{2}, \frac{q^2-q-2}{2})$	$\mathbb{Z}_q^n$	✓	$(q, \frac{q-1}{2}, \frac{q-3}{4})$ DS, $q \equiv 3 \pmod 4$	Paley difference set
$(q^n, kq, kq-1)$	$\mathbb{Z}_q^n$	✓	$(q,k,k-1)$ DF, $q \equiv 1 \pmod k$	Wilson, 1972
$(q^n, kq, \frac{kq-1}{2})$	$\mathbb{Z}_q^n$	✓	$(q,k,\frac{k-1}{2})$ DF, $q \equiv 1 \pmod k$, q,k odd	Wilson, 1972
$(q^n, kq, \frac{k(kq-1)}{k-1})$	$\mathbb{Z}_q^n$	✓	(q,k,k) DF, $q \equiv 1 \pmod{k-1}$	Wilson, 1972
$(q^n, kq, \frac{k(kq-1)}{2(k-1)})$	$\mathbb{Z}_q^n$	✓	$(q,k,\frac{k}{2})$ DF, $q \equiv 1 \pmod{k-1}$	Wilson, 1972

First try:

- We know:
- $(v, k, 1)$ difference family $\mathcal{F}$ in G $\quad\Rightarrow\quad$ 2-$(v, k, 1)$ design with $(\mathcal{P}, \mathcal{B})$
- $\mathcal{F} = \{D_1, \ldots, D_t\}$ in G $\quad\Rightarrow\quad$ $\mathcal{B} = \{B_i = D_i + g \,:\, 1 \leq i \leq n, g \in G\}$
- Possible idea: Choose blocks $D_1, \ldots, D_t$ such that $\sum_{x \in D_i} x = 0$
- We hope:

$$\sum_{x \in B_i} x = \sum_{x \in D_i + g} x = 0$$

- Unfortunately, this is not true. $\Rightarrow\Leftarrow$

Theorem (Buratti, A.N., 202?)

If $k \not\equiv 2 \pmod 4$ and $k \neq 2^n \cdot 3$, there are infinitely many values of v for which there exists a strictly additive 2-$(v, k, 1)$ design.

Few ideas from the proof (1).

- $[k \not\equiv 2 \pmod 4)]$ $\quad$ G abelian group od order k such that $\sum_{x \in G} = 0$
- If you can construct $(kp^n, k, k, 1)$ DF in $G \times \mathbb{F}_{p^n}$ relative to $G \times \{0\}$, p a prime divisor of k:

$$\Delta D_1 \cup \cdots \cup \Delta D_t = G \times \mathbb{F}_{q^n} \setminus G \times \{0\}$$

- such that

$$\sum_{x \in D_i} x = 0$$

- then we have:

$$\sum_{x \in D_i + g} x = 0 \text{ and } \sum_{x \in G \times \{y\}} x = 0$$

- We get a Steiner design with $\mathcal{B} = \{D_i + g\} \bigcup \{G \times \{y\}\}$

Theorem (Buratti, A.N., 202?)

If $k \not\equiv 2 \pmod 4$ and $k \neq 2^n \cdot 3$, there are infinitely many values of v for which there exists a strictly additive 2-$(v, k, 1)$ design.

Few ideas from the proof (2).

- Does such DF exists?
- $[k \neq 2^n \cdot 3]$ It can be constructed from (k, k, λ) strong DF in G such that

$$\Delta C_1 \cup \cdots \cup \Delta C_s = \lambda G \text{ and } \sum_{x \in C_i} x = 0$$

□

- $v = k \cdot p^n$, is huge, p prime divisor of k

Theorem (Buratti, A.N., 202?)
If $k \not\equiv 2 \pmod 4$ and $k \neq 2^n \cdot 3$, there are infinitely many values of v for which there exists a strictly additive 2-$(v, k, 1)$ design.

Constructing examples is computationally hard.

k	3	4	5
	AG$(n, 3)$	AG$(n, 4)$	AG$(n, 5)$

k	6	7	8	9	10
	$2^1 \cdot 3$	AG$(n, 7)$	AG$(n, 8)$	AG$(n, 9)$	$2 \pmod 4$

k	11	12	13	14	15
	AG$(n, 11)$	$2^2 \cdot 3$	AG$(n, 13)$	$2 \pmod 4$	?

- $v = 15 \cdot 5^n, n \geq 10^7$

Thank you for your attention!