Riemann–Hilbert problem on the torus: A case study

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Joint work with Gerald Teschl

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Der Wissenschaftsfonds.

Aim of this talk

Goal: Case study of a R-H problem on a torus

R-H problem: Genus one KdV R-H problem (Its–Matveev solution)

Application: Deift–Zhou analysis of KdV solutions

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Theorem (Akhiezer, Dubrovin, Its, Matveev)

8eX0@}n]X0nh0§}_H »j_§0]X9q ne«§_nj §n §\0 bL- 0x«9§_nj °_§\ q0H§}«h

$$[N_0;N_1] \; [\; [N_2;N_3] \; [\qquad [\; [N_{2X};\; 1\,]$$

H9j @0 L0 H}_@0L 0²qe_H_§e´ _j §0}h nT §\0 `9Hn@_ §\0§9 T«jH§_nj }0e9§0L §n §\0 \´q0}0ee_q§_H |_0h9jj «}T9H0 °_§\ «qq0} \00§ C n $[^{X}_{\_=0}[N_{2\_};N_{2\_+1}]$
rN_{2X+1} = *1 s 9jL }0e9§0L x«9j§_§_0 I*

$$x(^2;§) = 2@^2_2 \log\ _3(\,^{a\,2} + \,^{-}§ + K) + 2H$$

Belokolos, A. Bobenko, V. Enol'skii, A. Its and V. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations*, Springer Series in Nonlinear Dynamics, Berlin, 1994.

Aim of this talk

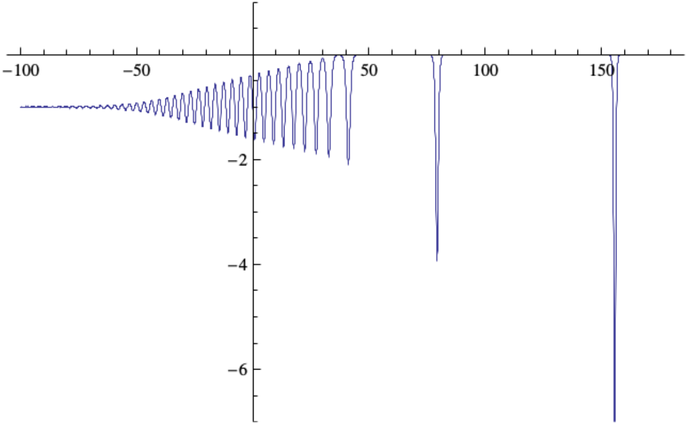

Figure: KdV solution with steplike initial data

Outline of the talk

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Genus one KdV R-H problem

Find a vector-valued function, holomorphic in $C\, n\, [\ \_H; \_H]$, $0 < 9 < H$,

$$h(c;{}^2;\S) = (h_1(c;{}^2;\S);\ h_2(c;{}^2;\S))$$

satisfying the **jump condition**,

$$h_+(c;{}^2;\S) = h_-(c;{}^2;\S)®(c;{}^2;\S); \quad c\ 2\ [\ \_H; \_H]$$

$$®(c;{}^2;\S) = \begin{cases} \begin{pmatrix} 0 & - \\ - & 0 \end{pmatrix}; & c\ 2\ [\_H; \_9]; \\ \begin{pmatrix} 0 & - \\ - & 0 \end{pmatrix}; & c\ 2\ [\ \_9;\ \_H]; \\ \begin{pmatrix} 0\,- & 0 \\ 0 & 0- \end{pmatrix}; & c\ 2\ [9;\ \_9]; \end{cases}$$

with $(^2;\S) = {}^{a\,2} + {}^{-}\S + K.$

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with $(^2;\S) = {}^{a\,2} + \bar{\ }\S + K.$

the **symmetry condition**,

$$h(\;\;c;{}^{2};\S) = h(c;{}^{2};\S)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}:$$

the **normalization condition**,

$$\lim_{c\;!\;1} h(c;{}^{2};\S) = (1;1):$$

and having at most **fourth root singularities** at $\underline{\;}9$:

$$h(c;{}^{2};\S) = m((c\quad\underline{\;}9)\quad^{1=4})$$

Question: How can we obtain a one gap solution $x({}^{2};\S)$ from $h(c;{}^{2};\S)$?

Answer: $x({}^{2};\S) = \lim_{c\;!\;1} 2c^{2}(h_1(c;{}^{2};\S)h_2(c;{}^{2};\S)\quad 1)$

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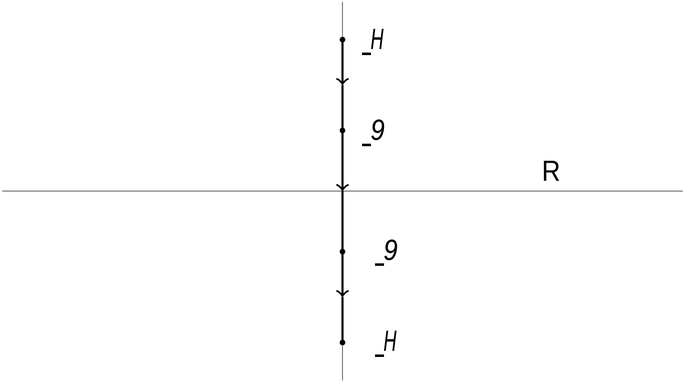

Figure: The jump contour

Outline of the talk

First step: scalar R-H problem

Find a scalar-valued function : C=([_H; _9] [[_9; _H]) ! C s.t.:

- $_+(c) = \_ \ (c); \quad c\ 2\ [\_9; \_H]$
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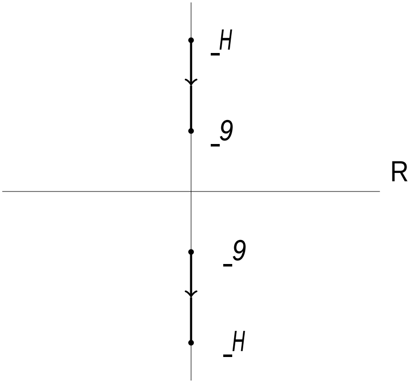
) Unique solution $(c) = \sqrt[4]{\frac{c^2+9^2}{c^2+H^2}}$



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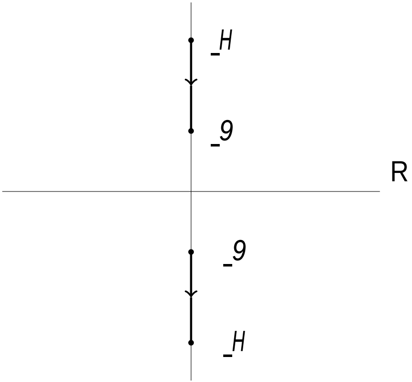

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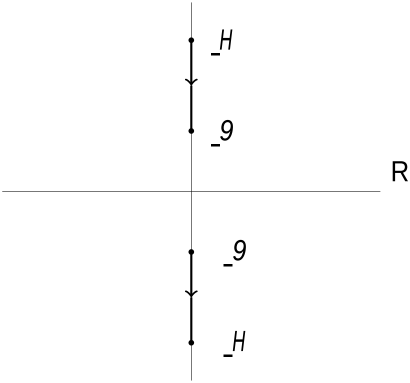

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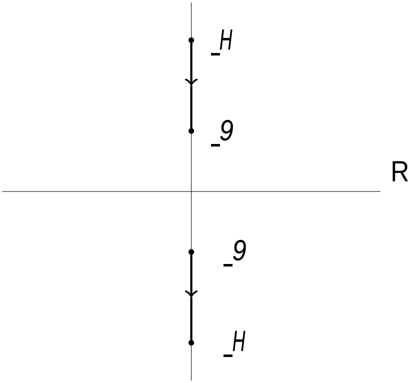

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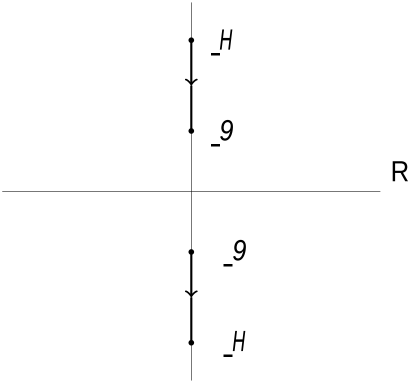

Second step: Conjugation

Define $j(c;{}^2;\S) := \ (c)\ ^1 h(c;{}^2;\S)$, where $(c) = \sqrt[4]{\frac{c^2+9^2}{c^2+H^2}}$

Then

$$j_+(c;{}^2;\S) = j\ (c;{}^2;\S)©(c;{}^2;\S); \quad c\ 2\ [\ \_H;\_H]$$

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In particular $j_{1;}\ (c;{}^2;\S) = j_{2;}\ (c;{}^2;\S)$, for $c\ 2\ [\ \ \_H;\ \ \_9]\ [\ [\_9;\_H]$.

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$$j_+(c;{}^2;\S) = j\ \ (c;{}^2;\S)\circledR(c;{}^2;\S); \quad c\ 2\ [\ \ \_H;\_H]$$

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Second step: Conjugation

Moreover, j satisfies the **symmetry condition**,

$$j(\ c;{}^2;\S) = j(c;{}^2;\S)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}:$$

the **normalization condition**,

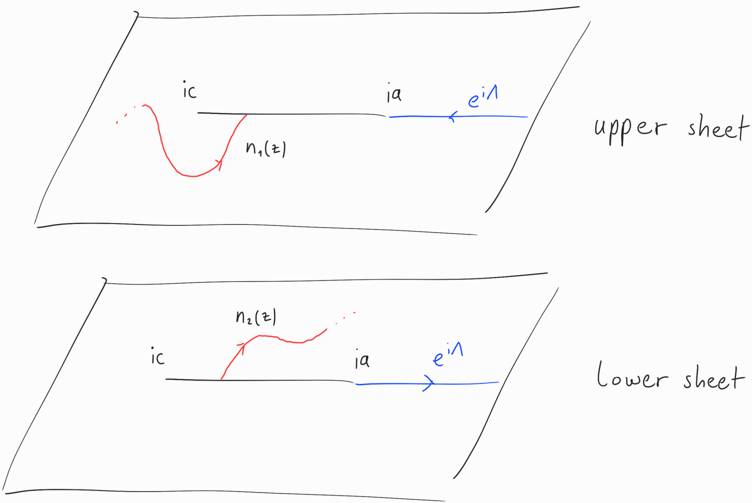
$$\lim_{c\ !\ 1} j(c;{}^2;\S) = (1;1):$$

and has at most **square root singularities** at $\underline{\ }9$:

$$j(c;{}^2;\S) = m((c\ \ \underline{\ }9)^{\ 1=2})$$

Second step: Conjugation

$$j_{1;}\ (c;{}^{2};\S) = j_{2;}\ (c;{}^{2};\S); \qquad c\ 2\ [\ \_H;\ \ \_9]\ [\ [\_9;\ \_H]$$



Second step: Conjugation

Define a function i on $\mathbf{X}$:

$$i((c;+)) = j_1(c); \quad \text{on the \textbf{upper} sheet}$$
$$i((c;\ )) = j_2(c); \quad \text{on the \textbf{lower} sheet}$$

$i(q)$, q 2 $\mathbf{X}$ is a holomorphic function with a jump on ^{e} $\mathbf{X}$:

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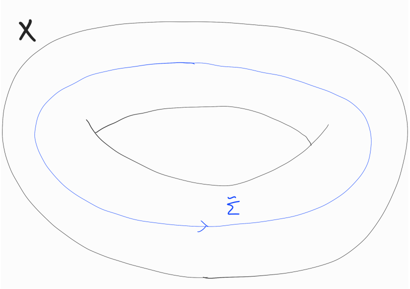
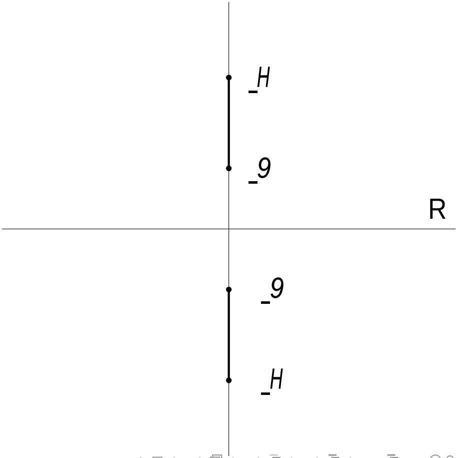


$$i_+(q) = i\ (q)0\text{-}\ ; \qquad q\ 2\ \text{e}$$

Isomorphism between **X** and C=

We now construct an explicit bijection from **X** to the torus C= :

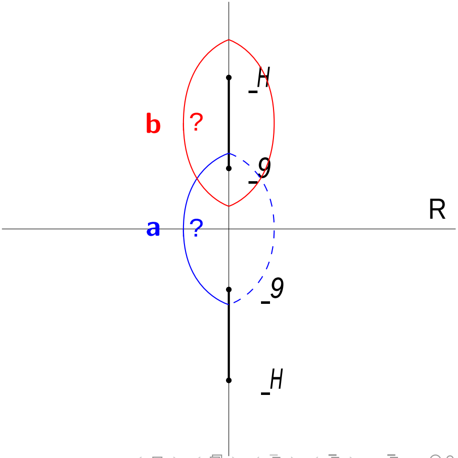


Isomorphism between X and C=

We now construct an explicit bijection from **X** to the torus C= :

We define a homological basis **a**, **b**:

The cycles **a**, **b** describe fully the topology of **X**:



L2ti r2 /2}M2 i?2 MQ`K HBx2/ ?QH QKQ`T?B+ /B z2`2MiB H QM

$$/! := p \frac{/}{(k + + k)(k + k)} ; \quad 2Cn([B; B][[B; B])$$

$$rBi?= R p \frac{/}{(k + + k)(k + k)} R) R /! = RX$$

.2}M2 MQr Ui?2 ? H7@T2`BQ/` iBQV

Z

:= /! 2 B+ :

#

L 2 t i r 2 / 2 } M 2 i ? 2 M Q ` K H B x 2 / ? Q H / Q K Q M T ? B + / B

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$$r B i ? := R \; p\frac{/}{(k+ +k)(k+ k)} \; R) \; R \; /! = R X$$

. 2 } M 2 M Q r U i ? 2 ? H 7 @ T 2 ` B Q / ` i B Q V

$$Z := \underset{\#}{} /! \; 2 \; B_+ :$$

> pBMr;2 + M MQr /2}M2 iiB?+22 H

:= f K+ M2 Cj K; M2 Zg

M/ i?2# 2H K T

$$: s! \; C= \; ; \quad T7! \, Z_{B+}^{T} \, /!:$$

h?2 #2H KBTb M #B?QHQKQ`T?BbK U?QHQKQ` s MC= X

) s' C= b _B2K MM bm`7 +2bX

> p B Mr; 2 + M M Q r / 2 } H M 2 i i B ? + 2 2

:= f K+ M 2 Cj K; M 2 Zg

M / i ? 2 # 2 H K T

$$: s! \quad C= \; ; \quad T7! \sum_{B+}^{Z\ T} /!:$$

h ? 2 # 2 H K B T b M # B ? Q H Q K Q ` T ? B b K U ? Q H Q K Q ` s M / C= X

) s ' C= b _B 2 K M M b m ` 7 + 2 b X

> pBMr;2 + M MQr /2}M2 iiB?+22

:= f K+ M2 Cj K; M2 Zg

M/ i?2# 2H K T

: s! C= ; T7! Z T B+ /!:

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. 2 } M 2 ((T)) := L(T) =) K(F) = (1((F); 1((F))

h ? 2 M(x+ R) = 1(x) M / i ? 2 7 Q H H Q r B M ; `2 2 [m B p H 2

$$K_+(F) = K(F) \frac{2^B}{y} \frac{y}{2^B} \quad 1(x+) = 1(x)\,2^B$$

$$K(F) = K(F) \frac{y}{R} \frac{R}{y} \quad 1\left(x+ \frac{R}{k}\right) = 1(x)$$

$$\lim_{F:1} K(F) = (R; R) \quad 1\left(\frac{R}{9}\right) = R$$

7 Q m `i ? `Q Q i b B M F;=m HB(B i B 2Q bH 21 x= $\frac{+}{k}$; $\frac{R+}{k}$:

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K+ (F) = K (F) 2^B y / y 2^B () 1(x+) = 1(x) 2^B

K(F) = K(F) y/R R/y () 1(x+ R/k) = 1(x)

lim_{F→1} K(F) = (R; R) () 1(R/9) = R

7 Q m ` i ? ` Q Q i b B M F = m H B ( ) B i E 2 Q b H 2 i x = + i; R+/k k :

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.B+iBQM `v #2Kir22M

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7 Q m ` i ? ` Q Q i b B M F = m H B ( ) B i B T 2 Q b H 2 i x = $\frac{+;}{k} \frac{R+}{k}$:

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7 Q m ` i ? ` Q Q i b B M F= m H B () B i B T 2 Q b H 2 i b = $\frac{i}{k}$; $\frac{R+}{k}$:

G2KK

G2i 6 #2 K2`QKQ`T?B+ 7mM+iBQM b iBb7vBM

$$6(x+R) = 6(x); \qquad 6(x+\;) = 6(x)\,2^{B}$$

h?2M

$$6(x) = + \prod_{B\;R}^{M} \frac{j(x \quad x_B \quad E)}{j(x \quad T_B \quad E)}$$

r?2`2=E$\frac{R}{k}$ M/j(x) = $\sum_{M2Z}$ exp(M^{k} + kM)x B B b i?2 C +Q

i?2i 7mM+iBQM X$_B$ M?/2TxQ2H Q b b b iBb7v

$$\sum_{B\;R}^{M} X_B \qquad \sum_{B\;R}^{M} T_B = \;:$$

*QMbi`m+iBQM Q7

G2KK

G2i 6 #2 K2`QKQ`T?B+ 7mM+iBQM b iBb7vBM

$$6(x+R) = 6(x); \qquad 6(x+\ ) = 6(x)\,2^{B}$$

h?2M

$$6(x) = + \prod_{B R}^{M} \frac{j(x \quad x_B \quad E)}{j(x \quad T_B \quad E)}$$

r?2`2 $E = \frac{R}{k}$ M / $j(x) = \sum_{M2Z} \exp(M^k + kM)x$ B Bb i?2 C +Q

i?2i 7mM+iBQM X_B M/ 2TxQ2H X_B Q7 6 b iBb7v

$$\sum_{B R}^{M} x_B \qquad \sum_{B R}^{M} T_B = \ :$$

*QMbi`m+iBQM Q7 1

SQH2 +Q)M/TBR+BQTM = R/k

avKK2i`v +Q)M/B+BQM

Zm bBT2`BQ/B+Biv U)Dm+KV +QM/BiBQM

) 1(x) = ... - mMB[m2M2bb 7Q` 7`22

*QMbi`m+iBQM Q7

SQH2 +Q)M/BiBQM: $T_R = T_k$; $T = \frac{R}{k}$

avKK2i`v +Q)M/BiBQM: $x \mapsto x + \frac{R}{k}$

Zm bBT2`BQ/B+Biv U)DmK TV +Q)M/BiBQM

$$1(x) = \frac{j(x + \frac{R}{k})\, j(x - \frac{R}{k})\, j(\frac{R}{9})^{k}}{j(x + \frac{R}{k})\, j(x)\, j(\frac{R}{9} - \frac{R}{k})\, j(\frac{R}{9} - \frac{R}{k})}$$ - mMB[m2M2bb 7Q` 7`2

*QMbi`m+iBQM Q7

SQH2 +QM/BiBQM: $\mathcal{R}$ ... $\frac{\mathcal{R}}{k}$

avKK2i`v +QM/BiBQM: x ... $\frac{\mathcal{R}}{k}$

Zm bBT2`BQ/B+Biv U)Dm+KTV +QM/BiBQM

$) \quad 1(x) = \frac{j(x - \frac{\mathcal{R}}{k} + \frac{\mathcal{R}}{k})\, j(x - \frac{}{k})\, j(\frac{\mathcal{R}}{9})^{k}}{j(x + \frac{\mathcal{R}}{k})\, j(x)\, j(\frac{\mathcal{R}}{9} - \frac{}{k})\, j(\frac{\mathcal{R}}{9} - \frac{}{k})}$ - mMB[m2M2bb 7Q` 7`22

Construction of N

Pole condition) $q_1 = \frac{}{2}; q_2 = \frac{1+}{2}$

Symmetry condition) $¶_2 = ¶_1 + \frac{1}{2}$

Quasiperiodicity (jump) condition) $¶_1 + ¶_2 \quad q_1 \quad q_2 =$

) $N(¶) = \frac{{}_3(¶ \quad \frac{}{2} + \frac{1}{2})\ {}_3(¶ \quad \frac{}{2})\ {}_3(\frac{1}{4})^2}{{}_3(¶ + \frac{1}{2})\ {}_3(¶)\ {}_3(\frac{1}{4} \quad \frac{}{2})\ {}_3(\quad \frac{1}{4} \quad \frac{}{2})}$, uniqueness for free

References

- I. Egorova, Z. Gladka, V. Kotlyarov, G. Teschl, *dnjX]§_h0 9 ´hq§n§_H Tn} §\0 bn}§0°0X]L0 -}_0 0x«9§_nj °_§\ §0qe_c0 _j_§_9e L9§9*. Nonlinearity **26**, 1839–1864 (2013).
- I. Egorova, M. Piorkowski, G. Teschl, *mj ®0H§n} 9jL h9§}_² |_0h9jj][_e@0}§ q}n@e0h Tn} bL- \nHc °9®0* , arXiv:1907.09792
- M. Girotti, T. Grava, R. Jenkins and T.-R. McLaughlin, *|_Xn«}n« 8 ´hq§n§_H nT 9 bL- ~ne_§nj W9* , Comm. Math. Phys, (2021).
- M. Piorkowski, *p9}9h0§}_² q}n@e0h Tn} §\0 bn}§0°0XŚL0 -}_0 0x«9§_nj °_§\ §0qe_c0 _j_§_9e L9§9*, arXiv:1908.11340
- M. Piorkowski and G. Teschl, *8 H9e9} |_0h9jjŚ[_e@0}§ q}n@e0h nj §\0 §n}« I 8qqe_H9§_nj §n §\0 bL- 0x«9§_nj*, arXiv:2106.10948.

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