Control of eigenfunctions on negatively curved surfaces

Semyon Dyatlov (MIT)

June 22, 2021
This talk presents a recent result in **quantum chaos**

**Central ingredient:** fractal uncertainty principle (FUP)

No function can be localized in both position and frequency near a fractal set

**Using tools from**
- Microlocal analysis (classical/quantum correspondence)
- Hyperbolic dynamics (classical chaos)
- Fractal geometry
- Harmonic analysis
This talk presents a recent result in quantum chaos

Central ingredient: fractal uncertainty principle (FUP)

No function can be localized in both position and frequency near a fractal set

Using tools from
- Microlocal analysis (classical/quantum correspondence)
- Hyperbolic dynamics (classical chaos)
- Fractal geometry
- Harmonic analysis
Control of eigenfunctions

- \((M, g)\) negatively curved surface
- Geodesic flow \(\varphi_t : T^*M \to T^*M\) is a standard model of classical chaos
- Eigenfunctions of the Laplacian \(-\Delta_g\) studied by quantum chaos

\[ (-\Delta_g - \lambda^2)u = 0, \quad \|u\|_{L^2} = 1 \]

Theorem 1

Let \(\Omega \subset M\) be an arbitrary nonempty open set. Then

\[ \|u\|_{L^2(\Omega)} \geq c > 0 \]

where \(c\) depends on \(M, \Omega\) but not on \(\lambda\)

Constant curvature: D–Jin '18, using D–Zahl '16 and Bourgain–D '18
Variable curvature: D–Jin–Nonnenmacher '19, using Bourgain–D '18
Control of eigenfunctions

- $(M, g)$ negatively curved surface
- Geodesic flow $\phi_t : T^*M \to T^*M$ is a standard model of classical chaos
- Eigenfunctions of the Laplacian $-\Delta_g$ studied by quantum chaos

$$(-\Delta_g - \lambda^2)u = 0, \quad \|u\|_{L^2} = 1$$

**Theorem 1**

Let $\Omega \subset M$ be an arbitrary nonempty open set. Then

$$\|u\|_{L^2(\Omega)} \geq c > 0$$

where $c$ depends on $M, \Omega$ but not on $\lambda$

Constant curvature: D–Jin ’18, using D–Zahl ’16 and Bourgain–D ’18

Control of eigenfunctions

- $(M, g)$ negatively curved surface
- Geodesic flow $\varphi_t : T^*M \to T^*M$ is a standard model of classical chaos
- Eigenfunctions of the Laplacian $-\Delta_g$ studied by quantum chaos

$$(-\Delta_g - \lambda^2)u = 0, \quad \|u\|_{L^2} = 1$$

**Theorem 1**

Let $\Omega \subset M$ be an arbitrary nonempty open set. Then

$$\|u\|_{L^2(\Omega)} \geq c > 0$$

where $c$ depends on $M, \Omega$ but not on $\lambda$

For bounded $\lambda$ the estimate follows from unique continuation principle

The new result is in the high frequency limit $\lambda \to \infty$
An illustration

Picture on the right courtesy of Alex Strohmaier, using Strohmaier–Uski ’12

Disk (Dirichlet b.c.)
Whitespace in the middle

Hyperbolic surface
No whitespace
A microlocal statement

We assume that \((M, g)\) has Anosov geodesic flow \(\phi_t : S^* M \to S^* M\).

\[
T(S^* M) = E_0 \oplus E_s \oplus E_u; \quad |d\phi_t(\rho)v| \leq Ce^{-\theta|t|} |v|, \quad \left\{ \begin{array}{ll}
 t \geq 0, & v \in E_s(\rho) \\
 t \leq 0, & v \in E_u(\rho)
\end{array} \right.
\]

Using a quantization procedure

\[
a \in C_c^\infty(T^* M) \quad \mapsto \quad \text{Op}_h(a) = a(x, \frac{h}{i} \partial_x) : L^2(M) \to L^2(M)
\]

\((-\Delta_g - \lambda^2)u = 0 \quad \implies \quad (-h^2\Delta_g - 1)u = 0, \quad h := \lambda^{-1}
\]

Theorem 1′

Assume that \(a|_{S^* M} \neq 0\). Then \(\exists C = C(a) : \) for all \(h \ll 1, u \in L^2(M)\)

\[
\|u\|_{L^2} \leq C \|\text{Op}_h(a)u\|_{L^2} + \frac{C \log(1/h)}{h} \|(-h^2\Delta_g - 1)u\|_{L^2}
\]
**A microlocal statement**

We assume that \((M, g)\) has **Anosov geodesic flow** \(\varphi_t : S^*M \to S^*M\)

\[
T(S^*M) = E_0 \oplus E_s \oplus E_u; \quad |d\varphi_t(\rho)v| \leq Ce^{-\theta|t||v|}, \begin{cases} t \geq 0, & v \in E_s(\rho) \\ t \leq 0, & v \in E_u(\rho) \end{cases}
\]

Using a quantization procedure

\[
a \in C^\infty_c(T^*M) \quad \mapsto \quad \operatorname{Op}_h(a) = a(x, \frac{h}{i}\partial_x) : L^2(M) \to L^2(M)
\]

\[
(-\Delta_g - \lambda^2)u = 0 \quad \Longrightarrow \quad (-h^2\Delta_g - 1)u = 0, \quad h := \lambda^{-1}
\]

**Theorem 1’**

Assume that \(a|_{S^*M} \neq 0\). Then \(\exists C = C(a) : \text{for all } h \ll 1, u \in L^2(M)\)

\[
\|u\|_{L^2} \leq C\|\operatorname{Op}_h(a)u\|_{L^2} + \frac{C\log(1/h)}{h}\|(-h^2\Delta_g - 1)u\|_{L^2}
\]
A microlocal statement

We assume that \((M,g)\) has Anosov geodesic flow \(\varphi_t : S^* M \to S^* M\)

\[
T(S^* M) = E_0 \oplus E_s \oplus E_u; \quad |d\varphi_t(\rho)v| \leq Ce^{-\theta|t|}|v|,
\left\{
\begin{array}{ll}
t \geq 0, & v \in E_s(\rho) \\
t \leq 0, & v \in E_u(\rho)
\end{array}
\right.
\]

Using a quantization procedure

\[
a \in C^\infty_c(T^* M) \mapsto \text{Op}_h(a) = a(x, \frac{h}{i} \partial_x) : L^2(M) \to L^2(M)
\]

\[
(-\Delta_g - \lambda^2)u = 0 \implies (-h^2\Delta_g - 1)u = 0, \quad h := \lambda^{-1}
\]

Theorem 1′

Assume that \(a|_{S^* M} \neq 0\). Then \(\exists C = C(a) : \text{for all } h \ll 1, \ u \in L^2(M)\)

\[
\|u\|_{L^2} \leq C \|\text{Op}_h(a)u\|_{L^2} + \frac{C \log(1/h)}{h} \|(-h^2\Delta_g - 1)u\|_{L^2}
\]
Theorem 1’

Assume that $a|_{S^*M} \neq 0$. Then $\exists C = C(a) :$ for all $h \ll 1$, $u \in L^2(M)$

$$\|u\| \leq C\|\text{Op}_h(a)u\| + \frac{C\log(1/h)}{h}\|(-h^2\Delta_g - 1)u\|$$

Remarks

- Implies Theorem 1: $a = a(x) \implies \text{Op}_h(a)u = au$
- Sharp: $a|_{S^*M} \equiv 0$, $(-h^2\Delta_g - 1)u = 0 \implies \|\text{Op}_h(a)u\| \leq Ch\|u\|$
- Cannot work for $O(h/\log(1/h))$ quasimodes: Brooks ’15, Eswarathasan–Nonnenmacher ’17, Eswarathasan–Silberman ’17

Applications

- Jin ’17: control/observability for Schrödinger equation
- Jin ’17, D–Jin–Nonnenmacher ’19: exponential energy decay for damped wave equation
- Datchev–Jin WIP, using Jin–Zhang ’17: a formula for $C(a)$
### Theorem 1’

Assume that \( a|_{S^*M} \neq 0 \). Then \( \exists C = C(a) : \) for all \( h \ll 1, \ u \in L^2(M) \)

\[
\|u\| \leq C \| \text{Op}_h(a)u \| + \frac{C \log(1/h)}{h} \|(-h^2 \Delta_g - 1)u\|
\]

### Remarks

- Implies Theorem 1: \( a = a(x) \implies \text{Op}_h(a)u = au \)
- Sharp: \( a|_{S^*M} \equiv 0, \ (-h^2 \Delta_g - 1)u = 0 \implies \| \text{Op}_h(a)u \| \leq Ch\|u\| \)
- Cannot work for \( O(h/\log(1/h)) \) quasimodes: Brooks ’15, Eswarathasan–Nonnenmacher ’17, Eswarathasan–Silberman ’17

### Applications

- **Jin ’17**: control/observability for Schrödinger equation
- **Jin ’17, D–Jin–Nonnenmacher ’19**: exponential energy decay for damped wave equation
- **Datchev–Jin WIP**, using Jin–Zhang ’17: a formula for \( C(a) \)
Semiclassical measures

Take a high frequency sequence of Laplacian eigenfunctions

\[ (-h_j^2 \Delta_g - 1) u_j = 0, \quad \| u_j \| = 1, \quad h_j \to 0 \]

We say \( u_j \) converges weakly to a measure \( \mu \) on \( T^* M \) if

\[ \forall a \in C_c^\infty (T^* M) : \quad \langle \text{Op}_{h_j}(a) u_j, u_j \rangle_{L^2} \to \int_{T^* M} a \, d\mu \quad \text{as} \quad j \to \infty \]

Call such limits \( \mu \) semiclassical measures

Basic properties

* \( \mu \) is a probability measure, \( \text{supp} \, \mu \subset S^* M \)
* \( \mu \) is invariant under the geodesic flow \( \varphi_t : S^* M \to S^* M \)
* Natural candidate: Liouville measure \( \mu_L \sim d\text{vol} \) (equidistribution)
* Natural enemy: delta measure \( \delta_\gamma \) on a closed geodesic (scarring)
Semiclassical measures

Take a high frequency sequence of Laplacian eigenfunctions

\[-h_j^2 \Delta_g - 1)u_j = 0, \quad \|u_j\| = 1, \quad h_j \to 0\]

We say \(u_j\) converges weakly to a measure \(\mu\) on \(T^*M\) if

\[\forall a \in C^\infty_c(T^*M) : \quad \langle \text{Op}_{h_j}(a)u_j, u_j \rangle_{L^2} \to \int_{T^*M} a \, d\mu \quad \text{as} \quad j \to \infty\]

Call such limits \(\mu\) semiclassical measures

Basic properties

- \(\mu\) is a probability measure, \(\text{supp } \mu \subset S^*M\)
- \(\mu\) is invariant under the geodesic flow \(\varphi_t : S^*M \to S^*M\)
- Natural candidate: Liouville measure \(\mu_L \sim d\text{vol}\) (equidistribution)
- Natural enemy: delta measure \(\delta_\gamma\) on a closed geodesic (scarring)
Semiclassical measures and Theorem 1

\[ (-h_j^2 \Delta_g - 1)u_j = 0, \quad \|u_j\| = 1, \quad h_j \to 0 \]

\[ \forall a \in C^\infty_c(T^*M) : \quad \langle \text{Op}_{h_j}(a)u_j, u_j \rangle_{L^2} \to \int_{T^*M} a \, d\mu \quad \text{as } j \to \infty \]

Theorem 1': \quad a|_{S^*M} \neq 0 \quad \implies \quad \| \text{Op}_{h_j}(a)u_j \| \geq c > 0

Theorem 1'':

Let \( \mu \) be a semiclassical measure on \( M \). Then \( \text{supp } \mu = S^*M \)

Brief overview of history

- Quantum Ergodicity [Shnirelman '74, Zelditch '87, Colin de Verdière '85, Z–Zworski '96]: \( \mu = \mu_L \) for density 1 sequence of \( u_j \)'s
- Quantum Unique Ergodicity conjecture [Rudnick–Sarnak '94]: \( \mu = \mu_L \) for all eigenfunctions, that is \( \mu_L \) is the only semiclassical measure. Proved in the arithmetic case [Lindenstrauss '06]
Semiclassical measures and Theorem 1

\[ (-h_j^2 \Delta_g - 1)u_j = 0, \quad \|u_j\| = 1, \quad h_j \to 0 \]

\[ \forall a \in C_c^\infty (T^*M) : \quad \langle \text{Op}_{h_j}(a)u_j, u_j \rangle_{L^2} \to \int_{T^*M} a \, d\mu \quad \text{as} \ j \to \infty \]

Theorem 1': \quad a|_{S^*M} \neq 0 \quad \implies \quad \|\text{Op}_{h_j}(a)u_j\| \geq c > 0

Theorem 1'':

Let \( \mu \) be a semiclassical measure on \( M \). Then \( \text{supp} \, \mu = S^*M \)

Brief overview of history

- Quantum Ergodicity [Shnirelman ’74, Zelditch ’87, Colin de Verdière ’85, Z–Zworski ’96]: \( \mu = \mu_L \) for density 1 sequence of \( u_j \)'s

- Quantum Unique Ergodicity conjecture [Rudnick–Sarnak ’94]: \( \mu = \mu_L \) for all eigenfunctions, that is \( \mu_L \) is the only semiclassical measure. Proved in the arithmetic case [Lindenstrauss ’06]
Semiclassical measures and Theorem 1

\[ (-h_j^2 \Delta_g - 1)u_j = 0, \quad \|u_j\| = 1, \quad h_j \to 0 \]

\[ \forall a \in C_c^\infty(T^*M) : \quad \langle \text{Op}_{h_j}(a)u_j, u_j \rangle_{L^2} \to \int_{T^*M} a \, d\mu \quad \text{as } j \to \infty \]

Theorem 1': \quad a|_{S^*M} \neq 0 \implies \|\text{Op}_{h_j}(a)u_j\| \geq c > 0

Theorem 1'':

Let \( \mu \) be a semiclassical measure on \( M \). Then \( \text{supp } \mu = S^*M \)

Brief overview of history, continued

- **Entropy bounds** [Anantharaman ’08, A–Nonnenmacher ’07, Rivière ’10, Anantharaman–Silberman ’13]:
  \[ H_{KS}(\mu) \geq c_{(M,g)} > 0, \text{ in particular } \mu \neq \delta_{\gamma} \]

- **Theorem 1''**: between QE and QUE and ‘orthogonal’ to entropy bound. There exist \( \mu \) with \( \text{supp } \mu \neq S^*M, \quad H_{KS}(\mu) > c_{(M,g)} \)
Main tool: fractal uncertainty principle (FUP)

No function can be localized in both position and frequency near a fractal set

Definition

Fix $\nu > 0$. A set $X \subset \mathbb{R}$ is $\nu$-porous up to scale $h$ if for each interval $I \subset \mathbb{R}$ of length $h \leq |I| \leq 1$, there is an interval $J \subset I$, $|J| = \nu |I|$, $J \cap X = \emptyset$

Example: mid-third Cantor set $C \subset [0, 1]$ is $\frac{1}{6}$-porous on scales 0 to 1

Theorem 2 [Bourgain–D’18]

Assume that $X, Y \subset \mathbb{R}$ are $\nu$-porous up to scale $h$. Then $\exists \beta = \beta(\nu) > 0$:

$$\|1_X(x)1_Y(\frac{h}{I} \partial_x)\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = \mathcal{O}(h^\beta) \quad \text{as } h \to 0$$

Note: enough that $X, Y$ be porous up to scales $h^{\alpha_X}, h^{\alpha_Y}, \alpha_X + \alpha_Y > 1$
Main tool: fractal uncertainty principle (FUP)

No function can be localized in both position and frequency near a fractal set

**Definition**

Fix $\nu > 0$. A set $X \subset \mathbb{R}$ is $\nu$-porous up to scale $h$ if for each interval $I \subset \mathbb{R}$ of length $h \leq |I| \leq 1$, there is an interval $J \subset I$, $|J| = \nu |I|$, $J \cap X = \emptyset$

**Example:** mid-third Cantor set $C \subset [0, 1]$ is $\frac{1}{6}$-porous on scales 0 to 1

**Theorem 2 [Bourgain–D ’18]**

Assume that $X, Y \subset \mathbb{R}$ are $\nu$-porous up to scale $h$. Then $\exists \beta = \beta(\nu) > 0$:

$$\|1_X(x)1_Y(\frac{h}{I} \partial_x)\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} = O(h^{\beta}) \quad \text{as } h \rightarrow 0$$

Note: enough that $X, Y$ be porous up to scales $h^{\alpha_X}, h^{\alpha_Y}, \alpha_X + \alpha_Y > 1$
Main tool: fractal uncertainty principle (FUP)

No function can be localized in both **position** and **frequency** near a fractal set

**Definition**

Fix $\nu > 0$. A set $X \subset \mathbb{R}$ is $\nu$-porous up to scale $h$ if for each interval $I \subset \mathbb{R}$ of length $h \leq |I| \leq 1$, there is an interval $J \subset I$, $|J| = \nu|I|$, $J \cap X = \emptyset$.

**Example:** mid-third Cantor set $C \subset [0, 1]$ is $\frac{1}{6}$-porous on scales 0 to 1.

**Theorem 2 [Bourgain–D’18]**

Assume that $X, Y \subset \mathbb{R}$ are $\nu$-porous up to scale $h$. Then $\exists \beta = \beta(\nu) > 0$:

$$\|1_X(x)1_Y\left(\frac{h}{i} \partial_x\right)\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = O(h^\beta) \quad \text{as } h \to 0$$

Note: enough that $X, Y$ be porous up to scales $h^{\alpha_X}, h^{\alpha_Y}, \alpha_X + \alpha_Y > 1$.
Main tool: fractal uncertainty principle (FUP)

No function can be localized in both position and frequency near a fractal set

Definition

Fix $\nu > 0$. A set $X \subset \mathbb{R}$ is $\nu$-porous up to scale $h$ if for each interval $I \subset \mathbb{R}$ of length $h \leq |I| \leq 1$, there is an interval $J \subset I$, $|J| = \nu |I|$, $J \cap X = \emptyset$

Example: mid-third Cantor set $\mathcal{C} \subset [0, 1]$ is $\frac{1}{6}$-porous on scales 0 to 1

Theorem 2 [Bourgain–D’18]

Assume that $X, Y \subset \mathbb{R}$ are $\nu$-porous up to scale $h$. Then $\exists \beta = \beta(\nu) > 0$:

$$\left\| 1_X(x) 1_Y \left( \frac{h}{i} \partial_x \right) \right\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = \mathcal{O}(h^\beta) \quad \text{as } h \to 0$$

Note: enough that $X, Y$ be porous up to scales $h^{\alpha_X}, h^{\alpha_Y}, \alpha_X + \alpha_Y > 1$
Proof of Theorem 1′

Theorem 1′
Assume that $a|_{S^* M} \not\equiv 0$. Then for all $h \ll 1$, $u \in L^2(M)$

$$\|u\|_{L^2} \leq C \|\text{Op}_h(a)u\| + \frac{C \log(1/h)}{h} \|(-h^2 \Delta_g - 1)u\|$$

Theorem 1′-weak
Assume that $a|_{S^* M} \not\equiv 0$. Then for all $h \ll 1$, $u \in L^2(M)$

$$(-h^2 \Delta_g - 1)u = 0 \implies \|u\| \leq C \log(1/h) \|\text{Op}_h(a)u\|$$

- To get rid of the $\log(1/h)$ term need to revise the argument in a way inspired by Anantharaman '08
- We present the proof for the variable curvature case but assume for simplicity $(M, g)$ is hyperbolic, i.e. has curvature $-1$
- WLOG $a \equiv 1$ on a nonempty open set $U \subset S^* M$ called the hole
Proof of Theorem 1′

**Theorem 1′**

Assume that \( a|_{S^* M} \neq 0 \). Then for all \( h \ll 1 \), \( u \in L^2(M) \)

\[
\|u\|_{L^2} \leq C \|\text{Op}_h(a)u\| + \frac{C \log(1/h)}{h} \|(-h^2 \Delta g - 1)u\|
\]

**Theorem 1′-weak**

Assume that \( a|_{S^* M} \neq 0 \). Then for all \( h \ll 1 \), \( u \in L^2(M) \)

\[
(-h^2 \Delta g - 1)u = 0 \implies \|u\| \leq C \log(1/h) \|\text{Op}_h(a)u\|
\]

- To get rid of the \( \log(1/h) \) term need to revise the argument in a way inspired by Anantharaman ’08
- We present the proof for the variable curvature case but assume for simplicity \((M, g)\) is hyperbolic, i.e. has curvature \(-1\)
- WLOG \( a \equiv 1 \) on a nonempty open set \( \mathcal{U} \subset S^* M \) called the hole
Proof of Theorem 1′

**Theorem 1′**

Assume that \( a|_{S^*M} \neq 0 \). Then for all \( h \ll 1 \), \( u \in L^2(M) \)

\[
\| u \|_{L^2} \leq C \| \text{Op}_h(a)u \| + \frac{C \log(1/h)}{h} \| (-h^2 \Delta_g - 1)u \|
\]

**Theorem 1′-weak**

Assume that \( a|_{S^*M} \neq 0 \). Then for all \( h \ll 1 \), \( u \in L^2(M) \)

\[
(-h^2 \Delta_g - 1)u = 0 \implies \| u \| \leq C \log(1/h) \| \text{Op}_h(a)u \|
\]

- To get rid of the \( \log(1/h) \) term need to revise the argument in a way inspired by Anantharaman ‘08
- We present the proof for the variable curvature case but assume for simplicity \((M, g)\) is hyperbolic, i.e. has curvature \(-1\)
- WLOG \( a \equiv 1 \) on a nonempty open set \( \mathcal{U} \subset S^*M \) called the hole
Proof of Theorem 1′

**Theorem 1′**
Assume that \( a|_{S^*M} \neq 0 \). Then for all \( h \ll 1, u \in L^2(M) \)

\[
\| u \|_{L^2} \leq C \| \text{Op}_h(a)u \| + \frac{C \log(1/h)}{h} \| (-h^2 \Delta_g - 1)u \|
\]

**Theorem 1′-weak**
Assume that \( a|_{S^*M} \neq 0 \). Then for all \( h \ll 1, u \in L^2(M) \)

\[
(-h^2 \Delta_g - 1)u = 0 \implies \| u \| \leq C \log(1/h) \| \text{Op}_h(a)u \|
\]

- To get rid of the \( \log(1/h) \) term need to revise the argument in a way inspired by Anantharaman ’08
- We present the proof for the variable curvature case but assume for simplicity \((M, g)\) is hyperbolic, i.e. has curvature \(-1\)
- WLOG \( a \equiv 1 \) on a nonempty open set \( \mathcal{U} \subset S^*M \) called the hole
Theorem 1′-weak
Assume that \( a \equiv 1 \) on a nonempty open \( \mathcal{U} \subset S^*M \). Then for \( h \ll 1 \)

\[ (-h^2 \Delta_g - 1)u = 0 \implies \|u\| \leq C \log(1/h) \| \text{Op}_h(a)u \| \]

- Write \( I = A_1 + A_\star \), \( A_1 = \text{Op}_h(a) \), \( \text{WF}_h(A_\star) \cap \mathcal{U} = \emptyset \)
- Wave propagator \( U(t) = e^{-it\sqrt{-\Delta_g}} \), \( U(t)u = e^{-it/h}u \)
- \( A(t) := U(-t)AU(t) \implies \|A_1(t)u\| = \|A_1u\| \)
- \( \implies u = A_\star(t)u + O(\|\text{Op}_h(a)u\|) \)
- Take \( N := \tau \log(1/h) \), \( \tau < 1 \), use the above for \( t = N, \ldots, -N \):

\[ A^- := A_\star(N) \cdots A_\star(1)A_\star(0), \quad A^+ := A_\star(0)A_\star(-1) \cdots A_\star(-N); \]
\[ \|u\| \leq \|A^- A^+ u\| + C \log(1/h) \| \text{Op}_h(a)u \| \]

- Theorem 1′-weak now follows from the key estimate

\[ \|A^- A^+\|_{L^2 \to L^2} = O(h^\beta), \quad \beta = \beta(\mathcal{U}) > 0 \]
Theorem 1′-weak

Assume that \(a \equiv 1\) on a nonempty open \(\mathcal{U} \subset S^*M\). Then for \(h \ll 1\)

\[ (-h^2 \Delta_g - 1)u = 0 \implies \|u\| \leq C \log(1/h) \|\text{Op}_h(a)u\| \]

- Write \(I = A_1 + A_\star\), \(A_1 = \text{Op}_h(a)\), \(WF_h(A_\star) \cap \mathcal{U} = \emptyset\)
- Wave propagator \(U(t) = e^{-it\sqrt{-\Delta_g}}\), \(U(t)u = e^{-it/h}u\)
- \(A(t) := U(-t)AU(t) \implies \|A_1(t)u\| = \|A_1u\| \implies u = A_\star(t)u + \mathcal{O}(\|\text{Op}_h(a)u\|)\)
- Take \(N := \tau \log(1/h), \tau < 1\), use the above for \(t = N, \ldots, -N\):
  \[
  A^- := A_\star(N) \cdots A_\star(1)A_\star(0), \quad A^+ := A_\star(0)A_\star(-1) \cdots A_\star(-N);
  \|u\| \leq \|A^- A^+ u\| + C \log(1/h) \|\text{Op}_h(a)u\|
  \]
- Theorem 1′-weak now follows from the key estimate
  \[
  \|A^- A^+\|_{L^2 \to L^2} = \mathcal{O}(h^\beta), \quad \beta = \beta(\mathcal{U}) > 0
  \]
Theorem 1’-weak

Assume that $a \equiv 1$ on a nonempty open $\mathcal{U} \subset S^* M$. Then for $h \ll 1$

$$(-h^2 \Delta_g - 1)u = 0 \implies \|u\| \leq C \log(1/h) \|\text{Op}_h(a)u\|$$

- Write $I = A_1 + A_\star$, $A_1 = \text{Op}_h(a)$, $\text{WF}_h(A_\star) \cap \mathcal{U} = \emptyset$
- Wave propagator $U(t) = e^{-it\sqrt{-\Delta_g}}$, $U(t)u = e^{-it/h}u$
- $A(t) := U(-t)AU(t) \implies \|A_1(t)u\| = \|A_1u\|$
  $$\implies u = A_\star(t)u + O(\|\text{Op}_h(a)u\|)$$
- Take $N := \tau \log(1/h)$, $\tau < 1$, use the above for $t = N, \ldots, -N$:
  $$A^- := A_\star(N) \cdots A_\star(1)A_\star(0), \quad A^+ := A_\star(0)A_\star(-1) \cdots A_\star(-N);$$
  $$\|u\| \leq \|A^- A^+ u\| + C \log(1/h) \|\text{Op}_h(a)u\|$$
- Theorem 1’-weak now follows from the key estimate
  $$\|A^- A^+\|_{L^2 \to L^2} = O(h^\beta), \quad \beta = \beta(\mathcal{U}) > 0$$
Theorem 1′-weak

Assume that \(a \equiv 1\) on a nonempty open \(\mathcal{U} \subset S^*M\). Then for \(h \ll 1\)

\[\left(-h^2\Delta_g - 1\right)u = 0 \implies \|u\| \leq C \log(1/h)\|\text{Op}_h(a)u\|\]

- Write \(I = A_1 + A_\ast\), \(A_1 = \text{Op}_h(a)\), \(\text{WF}_h(A_\ast) \cap \mathcal{U} = \emptyset\)
- Wave propagator \(U(t) = e^{-it\sqrt{-\Delta_g}}\), \(U(t)u = e^{-it/h}u\)
- \(A(t) := U(-t)AU(t) \implies \|A_1(t)u\| = \|A_1u\|\)
  \[\implies u = A_\ast(t)u + O(\|\text{Op}_h(a)u\|)\]
- Take \(N := \tau \log(1/h)\), \(\tau < 1\), use the above for \(t = N, \ldots, -N:\)
  \[A^- := A_\ast(N) \cdots A_\ast(1)A_\ast(0), \quad A^+ := A_\ast(0)A_\ast(-1) \cdots A_\ast(-N);\]
  \[\|u\| \leq \|A^- A^+ u\| + C \log(1/h)\|\text{Op}_h(a)u\|\]
- Theorem 1′-weak now follows from the key estimate
  \[\|A^- A^+\|_{L^2 \to L^2} = O(h^\beta), \quad \beta = \beta(\mathcal{U}) > 0\]
- $\text{WF}_h(A_\star) \cap \mathcal{U} = \emptyset$ where $\mathcal{U} \subset S^*M$ open nonempty, called the hole
- Need the key estimate $\|A^{-}A^{+}\|_{L^2 \to L^2} = O(h^\beta)$ where $N = \tau \log(1/h)$

$$A^{-} := A_\star(N) \cdots A_\star(1)A_\star(0), \quad A^{+} := A_\star(0)A_\star(-1) \cdots A_\star(-N)$$

- Egorov’s Theorem $\implies A^\pm$ microlocalized in $(\varphi_t = \text{geodesic flow})$

$$\Gamma^\pm(N) := \{\rho \in T^*M \mid \varphi_{\pm j}(\rho) \notin \mathcal{U} \quad \text{for all} \quad j = 0, 1, \ldots, N\}$$

$\Gamma^-(N), \ N = 0$ Hole (in white) $\Gamma^+(N), \ N = 0$ (using Arnold cat map model for the figures)
\begin{itemize}
  \item $WF_h(A_\star) \cap \mathcal{U} = \emptyset$ where $\mathcal{U} \subset S^*M$ open nonempty, called the \textit{hole}
  \item Need the key estimate $\|A^- A^+\|_{L^2 \rightarrow L^2} = O(h^\beta)$ where $N = \tau \log(1/h)$
    \begin{align*}
    & A^- := A_\star(N) \cdots A_\star(1)A_\star(0), \quad A^+ := A_\star(0)A_\star(-1) \cdots A_\star(-N) \\
    \end{align*}
  \item Egorov’s Theorem $\implies A^\pm$ microlocalized in $(\varphi_t = \text{geodesic flow})$
    \begin{align*}
    & \Gamma^\pm(N) := \{\rho \in T^*M \mid \varphi_{\mp j}(\rho) \notin \mathcal{U} \text{ for all } j = 0, 1, \ldots, N\}
    \end{align*}
\end{itemize}

\[ \Gamma_-(N), \ N = 0 \quad \text{Hole (in white)} \quad \Gamma_+(N), \ N = 0 \]

(using Arnold cat map model for the figures)
\begin{itemize}
\item $\text{WF}_h(A_\star) \cap \mathcal{U} = \emptyset$ where $\mathcal{U} \subset S^*M$ open nonempty, called the \textbf{hole}
\item Need the key estimate $\|A^{-}A^{+}\|_{L^2 \to L^2} = \mathcal{O}(h^\beta)$ where $N = \tau \log(1/h)$
\begin{align*}
A^{-} &:= A_\star(N) \cdots A_\star(1)A_\star(0), \\
A^{+} &:= A_\star(0)A_\star(-1) \cdots A_\star(-N)
\end{align*}
\item Egorov’s Theorem $\implies A^{\pm}$ microlocalized in ($\varphi_t = \text{geodesic flow}$)
\begin{align*}
\Gamma^{\pm}(N) &:= \{ \rho \in T^*M \mid \varphi_{\pm j}(\rho) \notin \mathcal{U} \text{ for all } j = 0, 1, \ldots, N \}
\end{align*}
\end{itemize}

\begin{tabular}{ccc}
$\Gamma_-(N), \ N = 0$ & Hole (in white) & $\Gamma_+(N), \ N = 0$ \\
\end{tabular}

(using Arnold cat map model for the figures)
Proof

- \( \text{WF}_h(A_\star) \cap U = \emptyset \) where \( U \subset S^*M \) open nonempty, called the hole
- Need the key estimate \( \| A^- A^+ \|_{L^2 \to L^2} = O(h^\beta) \) where \( N = \tau \log(1/h) \)
  \[
  A^- := A_\star(N) \cdots A_\star(1)A_\star(0), \quad A^+ := A_\star(0)A_\star(-1) \cdots A_\star(-N)
  \]
- Egorov’s Theorem \( \implies \) \( A^\pm \) microlocalized in \( (\varphi_t = \text{geodesic flow}) \)
  \[
  \Gamma^\pm(N) := \{ \rho \in T^*M | \varphi_\mp j(\rho) \notin U \text{ for all } j = 0, 1, \ldots, N \}
  \]

\( \Gamma_-(N), N = 1 \) \hspace{1cm} \text{Hole (in white)} \hspace{1cm} \Gamma_+(N), N = 1 \)

(using Arnold cat map model for the figures)
Proof

- \( \text{WF}_h(A_\star) \cap \mathcal{U} = \emptyset \) where \( \mathcal{U} \subset S^*M \) open nonempty, called the hole
- Need the key estimate \( \|A^- A^+\|_{L^2 \to L^2} = O(h^\beta) \) where \( N = \tau \log(1/h) \)

\[
A^- := A_\star(N) \cdots A_\star(1)A_\star(0), \quad A^+ := A_\star(0)A_\star(-1) \cdots A_\star(-N)
\]

- Egorov’s Theorem \( \implies \) \( A^\pm \) microlocalized in \((\varphi_t = \text{geodesic flow})\)

\[
\Gamma^\pm(N) := \{ \rho \in T^*M \mid \varphi_{\mp j}(\rho) \notin \mathcal{U} \text{ for all } j = 0, 1, \ldots, N \}
\]

\( \Gamma_-(N), \ N = 2 \) \hspace{2cm} \text{Hole (in white)} \hspace{2cm} \Gamma_+(N), \ N = 2 \)

(using Arnold cat map model for the figures)
Proof

- $\text{WF}_h(A_\ast) \cap U = \emptyset$ where $U \subset S^*M$ open nonempty, called the hole
- Need the key estimate $\|A^- A^+\|_{L^2 \to L^2} = O(h^\beta)$ where $N = \tau \log(1/h)$
  
  \[
  A^- := A_\ast(N) \cdots A_\ast(1) A_\ast(0), \quad A^+ := A_\ast(0) A_\ast(-1) \cdots A_\ast(-N)
  \]
- Egorov’s Theorem $\implies A^\pm$ microlocalized in ($\varphi_t$ = geodesic flow)
  
  \[
  \Gamma^{\pm}(N) := \{\rho \in T^*M \mid \varphi_\mp j(\rho) \notin U \text{ for all } j = 0, 1, \ldots, N\}
  \]

\[\Gamma_-(N), \ N = 3 \quad \text{Hole (in white)} \quad \Gamma_+(N), \ N = 3\]

(using Arnold cat map model for the figures)
Proof

- $WF_h(A_\star) \cap \mathcal{U} = \emptyset$ where $\mathcal{U} \subset S^*M$ open nonempty, called the hole.
- Need the key estimate $\|A^- A^+\|_{L^2 \to L^2} = \mathcal{O}(h^\beta)$ where $N = \tau \log(1/h)$.
  \[ A^- := A_\star(N) \cdots A_\star(1) A_\star(0), \quad A^+ := A_\star(0) A_\star(-1) \cdots A_\star(-N) \]
- Egorov's Theorem $\implies A^\pm$ microlocalized in ($\varphi_t =$ geodesic flow)
  \[ \Gamma^\pm(N) := \{ \rho \in T^*M \mid \varphi_{\mp j}(\rho) \notin \mathcal{U} \quad \text{for all} \quad j = 0, 1, \ldots, N \} \]

\[ \Gamma_-(N), \ N = 4 \quad \text{Hole (in white)} \quad \Gamma_+(N), \ N = 4 \]

(using Arnold cat map model for the figures)
• $\text{WF}_h(A_\star) \cap U = \emptyset$ where $U \subset S^*M$ open nonempty, called the **hole**

• Need the key estimate $\|A^- A^+\|_{L^2 \to L^2} = O(h^\beta)$ where $N = \tau \log(1/h)$

\[
A^- := A_\star(N) \cdots A_\star(1) A_\star(0), \quad A^+ := A_\star(0) A_\star(-1) \cdots A_\star(-N)
\]

• Egorov’s Theorem $\implies$ $A^\pm$ microlocalized in $(\varphi_t = \text{geodesic flow})$

\[
\Gamma^\pm(N) := \{\rho \in T^*M \mid \varphi^{\mp j}(\rho) \notin U \text{ for all } j = 0, 1, \ldots, N\}
\]

\[\begin{array}{ccc}
\Gamma_-(N), \; N = 5 & \text{Hole (in white)} & \Gamma_+(N), \; N = 5 \\
\end{array}\]

(using Arnold cat map model for the figures)
Proof

- Key estimate: \( \| A^- A^+ \|_{L^2 \to L^2} = O(h^\beta) \), 
  \( A^\pm \) microlocalized on \( \Gamma^\pm(N) \), \( N = \tau \log(1/h) \)
- \( \Gamma^+ \) smooth in the unstable direction, porous up to scale \( h^\tau \) in the stable direction
- Same true for \( \Gamma^- \), switching stable/unstable
- The product \( A^- A^+ \) is not pseudodifferential
- Will use FUP to show the key estimate

**Challenges in variable curvature**

- Variable expansion rates of the flow \( \varphi_t \)
  \( \implies \) take a dynamically fine partition 
  \( A_* = A_2 + \cdots + A_L \) and put \( N = \text{local Ehrenfest time for each word} \)
- Stable/unstable foliations are not \( C^\infty \)
  \( \implies \) cannot make \( A^\pm \) pseudodifferential following D–Zahl '16
Proof

- Key estimate: $\| A^- A^+ \|_{L^2 \to L^2} = O(h^\beta)$, $A^\pm$ microlocalized on $\Gamma^\pm(N)$, $N = \tau \log(1/h)$
- $\Gamma^+$ smooth in the unstable direction, porous up to scale $h^\tau$ in the stable direction
- Same true for $\Gamma^-$, switching stable/unstable
- The product $A^- A^+$ is not pseudodifferential
- Will use FUP to show the key estimate

Challenges in variable curvature

- Variable expansion rates of the flow $\varphi_t$
  \[ \implies \text{take a dynamically fine partition} \]
  \[ A_\tau = A_2 + \cdots + A_L \text{ and put } N = \text{local Ehrenfest time for each word} \]
- Stable/unstable foliations are not $C^\infty$
  \[ \implies \text{cannot make } A^\pm \text{ pseudodifferential} \]
  following D–Zahl ’16
Key estimate: \( \| A^- A^+ \|_{L^2 \to L^2} = \mathcal{O}(h^\beta) \), 
\( A^\pm \) microlocalized on \( \Gamma^\pm(N) \), \( N = \tau \log(1/h) \)

\( \Gamma^+ \) smooth in the unstable direction, porous up to scale \( h^\tau \) in the stable direction

Same true for \( \Gamma^- \), switching stable/unstable

The product \( A^- A^+ \) is not pseudodifferential

Will use FUP to show the key estimate

---

Challenges in variable curvature

- Variable expansion rates of the flow \( \varphi_t \)
  \( \implies \) take a dynamically fine partition
  \( A_* = A_2 + \cdots + A_L \) and put \( N = \text{local Ehrenfest time} \) for each word

- Stable/unstable foliations are not \( C^\infty \)
  \( \implies \) cannot make \( A^\pm \) pseudodifferential following D–Zahl ’16
Proof

Reduction to FUP

\[ \| A^- A^+ \|_{L^2(M) \to L^2(M)} = O(h^\beta) \]

\[ \| 1_X(x) \chi_Y(\frac{h}{i} \partial_x) \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = O(h^\beta) \]

- Restrict to \( S^* M \), remove the flow direction: 2D \( \iff \) 1D
- Conjugate by a Fourier Integral Operator? But cannot straighten out the stable/unstable foliations simultaneously (and they are not \( C^\infty \))
Proof

Reduction to FUP

\[ \| A^- A^+ \|_{L^2(M) \to L^2(M)} = O(h^\beta) \quad \| \chi(x) \chi(x) \frac{h}{i} \partial_x \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = O(h^\beta) \]

- Restrict to \( S^* M \), remove the flow direction: 2D \( \iff \) 1D
- Conjugate by a Fourier Integral Operator? But cannot straighten out the stable/unstable foliations simultaneously (and they are not \( C^\infty \))
Cluster decomposition

- Replace $A^-$ by $\tilde{A}^-$ which microlocalizes to an $h^{1/6}$ neighborhood of $\Gamma^-$.
- Write $A^+ = \sum_j A^+_j$ where each $A^+_j$ microlocalizes to an $h^{2/3}$ neighborhood of some unstable leaf.
- $h^{1/6} \cdot h^{2/3} \gg h \implies B_j := \tilde{A}^- A^+_j$ are almost orthogonal:
  \[ \|B^*_j B'_j\|_{L^2 \to L^2}, \|B'_j B^*_j\|_{L^2 \to L^2} = O(h^\infty) \quad \text{when } |j - j'| \gg 1 \]
- By Cotlar–Stein enough to show
  \[ \max_j \|\tilde{A}^- A^+_j\|_{L^2 \to L^2} = O(h^\beta) \]
Cluster decomposition

- Replace $A^-$ by $\tilde{A}^-$ which microlocalizes to an $h^{1/6}$ neighborhood of $\Gamma^-$.
- Write $A^+ = \sum_j A_j^+$ where each $A_j^+$ microlocalizes to an $h^{2/3}$ neighborhood of some unstable leaf.
- $h^{1/6} \cdot h^{2/3} \gg h \implies B_j := \tilde{A}^- A_j^+$ are almost orthogonal:
  $$\|B_j^* B_{j'}\|_{L^2 \to L^2}, \|B_{j'} B_j^*\|_{L^2 \to L^2} = O(h^\infty) \quad \text{when } |j - j'| \gg 1$$
- By Cotlar–Stein enough to show
  $$\max_j \|\tilde{A}^- A_j^+\|_{L^2 \to L^2} = O(h^\beta)$$
Cluster decomposition

- Replace $A^-$ by $\tilde{\Xi}^-$ which microlocalizes to an $h^{1/6}$ neighborhood of $\Gamma^-$
- Write $A^+ = \sum_j A^+_j$ where each $A^+_j$ microlocalizes to an $h^{2/3}$ neighborhood of some unstable leaf
- $h^{1/6} \cdot h^{2/3} \gg h \implies B_j := \tilde{\Xi}^- A^+_j$ are almost orthogonal:
  \[
  \|B_j^* B_{j'}\|_{L^2 \rightarrow L^2}, \|B_{j'} B_j^*\|_{L^2 \rightarrow L^2} = O(h^\infty) \quad \text{when } |j - j'| \gg 1
  \]
- By Cotlar–Stein enough to show
  \[
  \max_j \|\tilde{\Xi}^- A^+_j\|_{L^2 \rightarrow L^2} = O(h^\beta)
  \]
Proof

Cluster decomposition

- Replace $A^-$ by $\tilde{A}^-$ which microlocalizes to an $h^{1/6}$ neighborhood of $\Gamma^-$
- Write $A^+ = \sum_j A_j^+$ where each $A_j^+$ microlocalizes to an $h^{2/3}$ neighborhood of some unstable leaf
- $h^{1/6} \cdot h^{2/3} \gg h \implies B_j := \tilde{A}^- A_j^+$ are almost orthogonal:
  \[
  \|B_j^* B_{j'}\|_{L^2 \to L^2}, \|B_{j'} B_j^*\|_{L^2 \to L^2} = O(h^\infty) \quad \text{when } |j - j'| \gg 1
  \]
- By Cotlar–Stein enough to show
  \[
  \max_j \|\tilde{A}^- A_j^+\|_{L^2 \to L^2} = O(h^\beta)
  \]
Cluster decomposition

- Replace $A^-$ by $\tilde{A}^-$ which microlocalizes to an $h^{1/6}$ neighborhood of $\Gamma^-$
- Write $A^+ = \sum_j A_j^+$ where each $A_j^+$ microlocalizes to an $h^{2/3}$ neighborhood of some unstable leaf
- $h^{1/6} \cdot h^{2/3} \gg h \implies B_j := \tilde{A}^- A_j^+$ are almost orthogonal:
  \[
  \|B_j^* B_{j'}\|_{L^2 \to L^2}, \|B_{j'} B_j^*\|_{L^2 \to L^2} = \mathcal{O}(h^\infty) \quad \text{when } |j - j'| \gg 1
  \]
- By Cotlar–Stein enough to show
  \[
  \max_j \|\tilde{A}^- A_j^+\|_{L^2 \to L^2} = \mathcal{O}(h^\beta)
  \]
Proof

Need \[ \| \tilde{A}^- A_j^+ \|_{L^2 \to L^2} = O(h^\beta); \quad \tilde{A}^- \leftrightarrow \tilde{\Gamma}^- := h^{1/6} \text{ neighborhood of } \Gamma^- , \]
\[ A_j^+ \leftrightarrow \Gamma_j^+ := \Gamma^+ \cap (h^{2/3}\text{-neighborhood of some unstable leaf } W_j) \]

As before, restrict to \( S^*M \) and remove the flow direction

Unstable foliation has \( C^{2-} \subset C^{3/2} \) regularity [Hurder–Katok ’90] \[ \implies \] construct \( C^\infty \) symplectomorphism \( \kappa \) to \( T^*\mathbb{R} \) s.t. unstable leaves \( h^{2/3}\text{-close to } W_j \) are mapped \( h\text{-close to horizontal lines} \)

Then \( \kappa(\Gamma_j^+) \subset \{ \xi \in \Omega^+ \} \), \( \kappa(\tilde{\Gamma}^- \cap \Gamma_j^+) \subset \{ x \in \Omega^- \} \) where \( \Omega^+, \Omega^- \subset \mathbb{R} \) are porous on scales up to \( h, h^{1/6} \)

Conjugate by an FIO quantizing \( \kappa \) to reduce to the FUP bound
\[ \| 1_{\Omega^-}(x) 1_{\Omega^+}(hDx) \|_{L^2 \to L^2} = O(h^\beta) \]
Proof

- Need $\|\tilde{A}^- A_j^+\|_{L^2 \to L^2} = O(h^\beta)$; $\tilde{A}^- \leftrightarrow \tilde{\Gamma}^- := h^{1/6}$ neighborhood of $\Gamma^-$, $A_j^+ \leftrightarrow \Gamma_j^+ := \Gamma^+ \cap (h^{2/3}\text{-neighborhood of some unstable leaf } W_j)$

- As before, restrict to $S^*M$ and remove the flow direction

- Unstable foliation has $C^2^- \subset C^{3/2}$ regularity [Hurder–Katok ’90] $\implies$ construct $C^\infty$ symplectomorphism $\kappa$ to $T^*\mathbb{R}$ s.t. unstable leaves $h^{2/3}\text{-close to } W_j$ are mapped $h\text{-close to horizontal lines}$

- Then $\kappa(\Gamma_j^+) \subset \{\xi \in \Omega^+\}$, $\kappa(\tilde{\Gamma}^- \cap \Gamma_j^+) \subset \{x \in \Omega^\}$

- Where $\Omega^+, \Omega^- \subset \mathbb{R}$ are porous on scales up to $h$, $h^{1/6}$

- Conjugate by an FIO quantizing $\kappa$ to reduce to the FUP bound $\|1_{\Omega^-}(x) 1_{\Omega^+}(hDx)\|_{L^2 \to L^2} = O(h^\beta)$
Proof

- Need $\| \tilde{A}^- A^+_j \|_{L^2 \to L^2} = O(h^\beta)$; $\tilde{A}^- \leftrightarrow \tilde{\Gamma}^- := h^{1/6}$ neighborhood of $\Gamma^-$, $A^+_j \leftrightarrow \Gamma^+_j := \Gamma^+ \cap (h^{2/3}\text{-neighborhood of some unstable leaf } W_j)$

- As before, restrict to $S^* M$ and remove the flow direction

- Unstable foliation has $C^{2-} \subset C^{3/2}$ regularity [Hurder–Katok ’90] $\implies$ construct $C^\infty$ symplectomorphism $\kappa$ to $T^* \mathbb{R}$ s.t. unstable leaves $h^{2/3}\text{-close to } W_j$ are mapped $h\text{-close to horizontal lines}$

- Then $\kappa(\Gamma^+_j) \subset \{ \xi \in \Omega^+ \}$, $\kappa(\tilde{\Gamma}^- \cap \Gamma^+_j) \subset \{ x \in \Omega^- \}$

- where $\Omega^+, \Omega^- \subset \mathbb{R}$ are porous on scales up to $h, h^{1/6}$

- Conjugate by an FIO quantizing $\kappa$ to reduce to the FUP bound

$$\| 1_{\Omega^-}(x) 1_{\Omega^+}(hD_x) \|_{L^2 \to L^2} = O(h^\beta)$$
Proof

需满足 \( \| \tilde{A}^+ \tilde{A}_j \|_{L^2 \to L^2} = O(h^{1/6}) \); \( \tilde{A}^- \leftrightarrow \tilde{\Gamma}^- := h^{1/6} \) 邻域 \( \Gamma^- \), \( A_j^+ \leftrightarrow \tilde{\Gamma}_j^+ := \Gamma^+ \cap (h^{2/3}\)-邻域\) 不稳定叶片

作为之前所述，限制到 \( S^*M \) 并去掉流动方向

不稳定的微分学有 \( C^{2-} \subset C^{3/2} \) 定义 \([\text{Hurder–Katok '90}]\) \( \implies \) 构造 \( C^\infty \) 同胚 \( \kappa \) 到 \( T^*\mathbb{R} \) 使得不稳定的叶片 \( h^{2/3}\)-接近 \( W_j \) 被映射为 \( h\)-接近水平线

则 \( \kappa(\tilde{\Gamma}_j^+) \subset \{ \xi \in \Omega^+ \} \), \( \kappa(\tilde{\Gamma}^- \cap \Gamma_j^+) \subset \{ \xi \in \Omega^- \} \)

其中 \( \Omega^+ \), \( \Omega^- \subset \mathbb{R} \) 是蓬松的不等式到 \( h \), \( h^{1/6} \)

通过 FIO 量化 \( \kappa \) 可以将之降到 FUP 界限

\[
\| 1_{\Omega^-}(x) 1_{\Omega^+}(hD_x) \|_{L^2 \to L^2} = O(h^\beta)
\]
Need \( \| \tilde{A}^- A^+_j \|_{L^2 \to L^2} = O(h^{\beta}) \); \( \tilde{A}^- \leftrightarrow \tilde{\Gamma}^- := h^{1/6} \) neighborhood of \( \Gamma^- \), \( A^+_j \leftrightarrow \Gamma^+_j := \Gamma^+ \cap (h^{2/3}\text{-neighborhood of some unstable leaf } W_j) \)

As before, restrict to \( S^*M \) and remove the flow direction

Unstable foliation has \( C^{2-} \subset C^{3/2} \) regularity [Hurder–Katok’90] \( \implies \) construct \( C^\infty \) symplectomorphism \( \kappa \) to \( T^*\mathbb{R} \) s.t. unstable leaves \( h^{2/3}\)-close to \( W_j \) are mapped \( h \)-close to horizontal lines

Then \( \kappa(\Gamma^+_j) \subset \{ \xi \in \Omega^+ \} \), \( \kappa(\tilde{\Gamma}^- \cap \Gamma^+_j) \subset \{ x \in \Omega^- \} \)

where \( \Omega^+, \Omega^- \subset \mathbb{R} \) are porous on scales up to \( h, h^{1/6} \)

Conjugate by an FIO quantizing \( \kappa \) to reduce to the FUP bound \( \| 1_{\Omega^-}(x)1_{\Omega^+}(hDx) \|_{L^2 \to L^2} = O(h^{\beta}) \)

To make the above arguments rigorous, use Egorov’s Theorem up to local Ehrenfest time (adapted from Rivière’10) and long logarithmic time propagation of Lagrangian states due to Anantharaman’08, Anantharaman–Nonnenmacher’07, Nonnenmacher–Zworski’09
Thank you for your attention!