Some Problems for a Mixed Type Equation Fractional Order with non-linear Loaded Term

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Note, that with intensive research on problems of optimal control of the agro-economical system, regulating the label of ground waters and soil moisture, it has become necessary to investigate BVPs for a loaded partial differential equations. Integral boundary conditions have various applications in thermo-elasticity, chemical engineering, population dynamics, etc. In this work we consider parabolic-hyperbolic type equation fractional order involving non-linear loaded term:

\[
0 = \begin{cases} 
  u_{xx} - c D_0^\alpha u + f_1(x, y; u(x, 0)), & \text{at } x > 0, \ y > 0 \\
  u_{xx} - u_{yy} + f_2(x, y; u(x + y, 0)), & \text{at } x > 0, \ y < 0 \\
  u_{xx} - u_{yy} + f_3(x, y; u_y(0, x + y)), & \text{at } x < 0, \ y > 0 
\end{cases} 
\]

where \( f_i(\cdot) \) are given functions, and \( c D_0^\alpha \) is Caputo operator:

\[
c D_0^\alpha f(y) = \frac{1}{\Gamma(1 - \alpha)} \int_0^y (y - z)^{-\alpha} f'(z)dz, \quad 0 < \alpha < 1, \quad (2)
\]
Considered domain

Let $\Omega \subset R^2$, be domain bounded with segments $B_2A_2$, $A_2A_1$ on the lines $x = l$, $y = h$ at $x > 0$, $y > 0$; and $A_1C_2$, $C_2B_1$ on the characteristics $x - y = l$, $x + y = 0$ of the Eq. (1) at $x > 0$, $y < 0$, also with the segments $B_1C_1$, $C_1B_2$ on the characteristics $y - x = h$, $x + y = 0$ of the Eq. (1) at $x < 0$, $y > 0$.

We denote as $\Omega_0$ parabolic part of the mixed domain $\Omega$, and hyperbolic parts through $\Omega_1$ at $x > 0$ and $\Omega_2$ at $x < 0$.
Non-local problem

In the domain $\Omega (\Omega = \Omega_1 \cup (B_1 B_2) \cup \Omega_0 \cup (B_1 A_1) \cup \Omega_2)$, we will investigate following NL problem. To find a solution $u(x, y)$ of Eq.(1) from the class:

$$W = \left\{ u \in C(\overline{\Omega}) \cap C^1 \left( \{ \overline{\Omega}_2 \setminus A_1 C_2 \} \cup \{ \overline{\Omega}_1 \setminus C_1 B_2 \} \right) \cap C^2 (\Omega_2 \cup \Omega_1) ; \right\}$$

satisfies boundary conditions:

$$u(l, t) = \varphi_1(y), \ 0 \leq y \leq h; \quad (3)$$

$$\frac{d}{dx} u \left( \frac{x}{2} ; -\frac{x}{2} \right) = a_1(x)u_y(x, 0) + a_2(x)u_x(x, 0) + a_3(x)u(x, 0) +$$

$$+ a_4(x), \ 0 \leq x < l; \quad (4)$$

$$\frac{d}{dy} u \left( -\frac{y}{2} ; \frac{y}{2} \right) = b_1(y)u_x(0, y) + b_2(y)u_y(0, y) + b_3(y)u(0, y) +$$

$$+ b_4(y), \ 0 \leq y < h; \quad (5)$$
and integral gluing condition:

\[
\lim_{y \to +0} y^{1-\alpha} u_y(x, y) = \lambda_1(x) u_y(x, -0) + \lambda_2(x) u_x(x, -0) + \\
\lambda_3(x) \int_0^x r_1(t) u(t, 0) dt + \lambda_4(x) u(x, 0) + \lambda_5(x), \quad 0 < x < l
\] (6)

\[
u_x(-0, y) = \mu_1(y) v_x(+0, y) + \mu_2(y) u_y(0, y) + \mu_3(y) \int_0^y r_2(t) u(0, t) dt + \\
\mu_4(y) u(0, y) + \mu_5(y), \quad 0 < y < h
\] (7)

where \(a_i(x), b_i(y), \ (i = 1, 2, 3, 4), \ \lambda_j(x), \ \mu_j(y), \ (j = 1, 2, 3, 4, 5), \ \varphi_1(y)\) are given functions, such that

\[
\sum_{k=1}^{4} \lambda_k^2(x) \neq 0, \ \sum_{k=1}^{3} a_k^2(x) \neq 0, \ \sum_{k=1}^{3} b_k^2(x) \neq 0 \ \text{and} \ \sum_{k=1}^{4} \mu_k^2(x) \neq 0.
\]
Remark.

The same problem with the local boundary and discontinuous gluing conditions for the Eq.(1) but without loaded term (i.e. for $f_i(x, y) \equiv 0 \ (i = 1, 2, 3)$) was investigated by B.Kadirkulov [1]. Formulated Problem in some special cases we will investigate with the same methods as in work [1], and a class of solution be expanded.

For instance

If one of the non-local conditions (4) and (5) changes to local condition (see fig.2), then the solution of the new problem just look in the class of functions:

$W_1 = \{ u : u \in C(\bar{\Omega}) \cap C^1 (\Omega_2 \cup B_1 C_2) \cap C^2 (\Omega_2 \cup \Omega_1); \ u_{xx}, c D_0^\alpha u \in C (\Omega_0) \}$ in case a),

and

$W_2 = \{ u : u \in C(\bar{\Omega}) \cap C^1 (\Omega_1 \cup B_1 C_1) \cap C^2 (\Omega_2 \cup \Omega_1); \ u_{xx}, c D_0^\alpha u \in C (\Omega_0) \}$ in case b).
Other problems

Figure: 2
The Method of investigation

On the certain conditions for the given functions, we can prove uniqueness of solution of the problem in cases a) and b), using by the method of integral energy. The existence of solution reduced to the Volterra and Fregholm types non-linear integral equations of the second kind respected to \( u_y(0, y) = \tau'_2(y) \) and \( u(x, 0) = \tau_1(x) \) accordingly.
Volterra type integral equations

We would like to note, that there are some problems for the Eq. (1), whose investigations will be reduced to the Volterra type non-linear integral equations (only), for instance:

![Diagram](image-url)
In cases $c.)$ and $d.)$ boundary condition (3) changed to the second boundary condition and the conditions (4) and (5) was replaced to the local conditions. In these cases, solution of the formulated problems, we are looking for in the class of functions:

\[ W_3 = \{ u : u \in C(\bar{\Omega}) \cap C^2(\Omega_2 \cup \Omega_1); \ u_x \in C(\Omega_0 \cup \overline{A_2B_2}); \ u_{xx}, CD_{oy}^\alpha u \in C(\Omega_0) \}. \]

Considering solution of the Cauchy problem (in hyperbolic domains) and a solution of the first BVP for the parabolic equation (see Eq.(1)), applying boundary conditions (4) and (5), taking into

\[
\lim_{y \to +0} y^{1-\alpha} u_y(x, y) = \nu_1^+(x), \quad u_x(+0, y) = \nu_2^+(y),
\]

\[
u_1^-(x), \quad u_y(x, -0) = \nu_1^-(x)
\]

we will find main functional relation:
Main functional relations

\[(2a_1(x) + 1)\nu_1^-(x) = (1 - 2a_2(x))\tau_1'(x) - 2a_3(x)\tau_1(x) - \frac{1}{2} \int_0^x f_2(\xi, x; \tau(\xi))d\xi - 2a_4(x),\] (8)

\[(2b_1(y) + 1)\nu_2(y) = (1 - 2b_2(y))\tau_2'(y) - 2b_3(y)\tau_2(y) - \frac{1}{2} \int_0^y f_3(\xi, y; \tau_y(\xi))d\xi - 2b_4(y),\] (9)

\[\tau_1''(x) - \Gamma(\alpha)\nu_1^+(x) + f_1(x, 0; \tau_1(x)) = 0,\] (10)

\[\nu_2^+(y) = -\int_0^y K(y - t)\tau_2'(t)dt + \Phi(y),\] (11)

where \(K(y - t)\) and \(\Phi(y)\) known functions.
Based on the gluing conditions (see (6) and (7))

\[
\nu_1^+(x) = \lambda_1(x)\nu_1^-(x) + \lambda_2(x)\tau_1'(x) + \lambda_3(x)\tau_1(x)dt + \lambda_4(x),
\]

(12)

\[
\nu_2^-(y) = \mu_1(y)\nu_2^+(y) + \mu_2(y)\tau_2(y) + \mu_3(y), \quad 0 < y < h,
\]

(13)

and considering (8)-(11), we receive an integral equation regarding 
\[ u(x, 0) = \tau_1(x) \] and \[ u_y(0, y) = \tau_2'(y). \]

Assume that the given functions are sufficiently smooth functions, and \( f_1(x, y, u(x, 0)) \in C(\Omega_0); f_2(x, y, u(x+y, 0)) \in C(\Omega_1); f_3(x, y, u_y(0, x+y)) \in C(\Omega_2), \) besides \( f_i(x, y, u) \) \((i = 1, 2, 3)\) satisfies Lipshits condition respected to the third argument, then we can prove unique solvability of the given integral equations.
Reference

Thank You!