On the Gross-Mansour-Tucker conjecture

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**Definition.** A *ribbon graph* $G$ is a surface with boundary represented as the union of two sets of closed topological discs called *vertex-discs* $\mathcal{V}(G)$ and *edge-ribbons* $\mathcal{E}(G)$, satisfying the following conditions:

- the vertex-discs and edge-ribbons intersect by disjoint line segments;
- each such line segment lies on the boundary of precisely one vertex-disc and precisely one edge-ribbon;
- every edge-ribbon contains exactly two such line segments.

**Example.**

![Diagram](image)
Partial duality of ribbon graphs.

$G = e = e' = G \{e\}$

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Partial duality of ribbon graphs.

Type \(pp\)

\[v_1 \neq v_2, f_1 \neq f_2\]

\(g \leftarrow \)

\(g + 1 \rightarrow\)

Type \(uu\)

\[v_1 = v_2, f_1 = f_2\]

Type \(pu\)

\[v_1 \neq v_2, f_1 = f_2\]

\(g \leftarrow \)

Type \(up\)

\[v_1 = v_2, f_1 \neq f_2\]

\(g \leftarrow \)

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Gross–Mansour–Tucker conjecture.


**GMT-conjecture.** *For any ribbon graph there is a subset of edges partial duality relative to which changes the genus.*

The *ribbon-join* (I. Moffatt) $G_1 \lor G_2$ is obtained by gluing together a vertex-disc of $G_1$ and a vertex-disc of $G_2$ along some arcs on their boundaries.

$$B_1 = \begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\end{array}
\end{array}, \quad B_2 = \begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\end{array}
\end{array}, \quad B_3 = \begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet} \\
\end{array}
\end{array}, \quad B_4 = \begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet} \\
\end{array}
\end{array}, \quad B_5 = \begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet} \\
\end{array}
\end{array}, \quad \ldots$$

**Theorem.** The genus of any partial dual to $B_{2n+1}$ is equal to $n$.

**Definition.** A connected ribbon graph $G$ **join-prime** if it cannot be represented as the ribbon-join of two graphs $G_1, G_2$ with at least one edge-ribbon each: $G \neq G_1 \lor G_2$. 
Main result.


**Theorem.** For any join-prime ribbon graph different from partial duals of $B_{2n+1}$, there are partial duals of different genus.

**Lemma.** Let $G$ be a one-vertex join-prime ribbon graph and $e \in E(G)$. Suppose that the genus of partial duals of $G$ stay the same, $g(G) = g(G^A)$ for all subsets $A \subseteq E(G)$. Then

1. $e$ is attached to different face-discs $f_1 \neq f_2$. That is $e$ has to be of Type up.

2. Any edge-ribbon interlaced with $e$ is attached to the same face-discs $f_1$ and $f_2$.

3. Any edge-ribbon not interlaced with $e$ is attached to a pair of face-discs different from $\{f_1, f_2\}$. 
Proof.

The non-orientable counterpart of the GMT conjecture.

- Maya Thompson (Royal Holloway University of London).

*The only non-orientable join-prime ribbon graph whose partial duals have the same Euler genus is the one-vertex ribbon graph with one twisted edge.*
THANK YOU!!!