Tutte's dichromate for signed graphs

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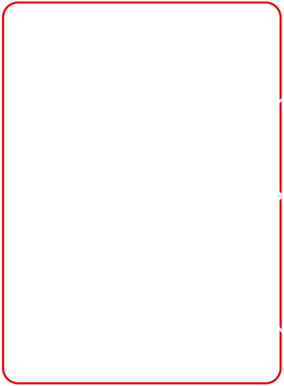
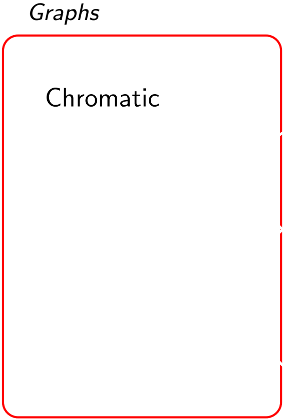
Overview

Graphs

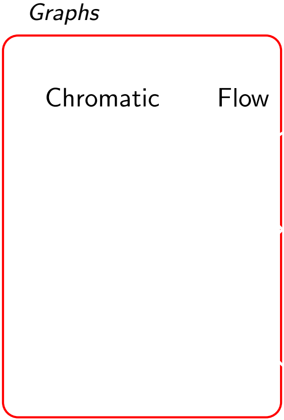


Overview



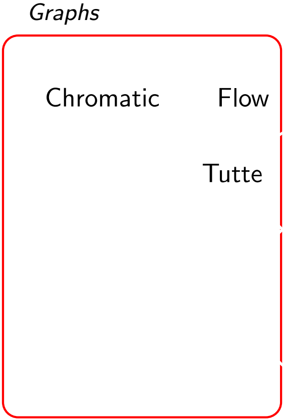
Chromatic polynomial: $\chi_\Gamma(n)$ number of proper colorings of graph Γ with n colors.

Overview



Flow polynomial: $\phi_\Gamma(n)$ number of nowhere-zero flows on graph Γ, values in $\mathbb{Z}_n \setminus \{0\}$.

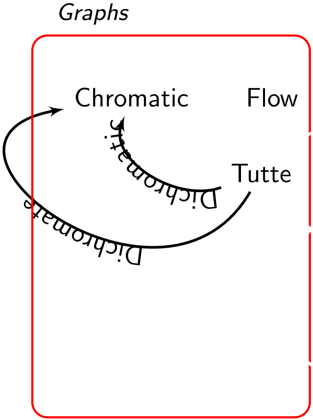
Overview



Tutte polynomial:

$$T_\Gamma(X, Y) = \sum_{A \subset E} (X-1)^{k(\Gamma \setminus A^c) - k(\Gamma)} (Y-1)^{|A| - |V| + k(\Gamma \setminus A^c)}$$

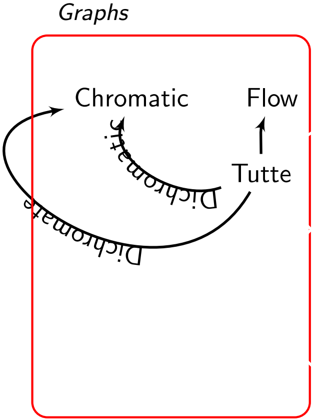
Overview



Tutte polynomial: Specializes to the chromatic polynomial

$$T_\Gamma(1-n,0) = (-1)^{|V|-k(\Gamma)} n^{k(\Gamma)} \chi_\Gamma(n)$$

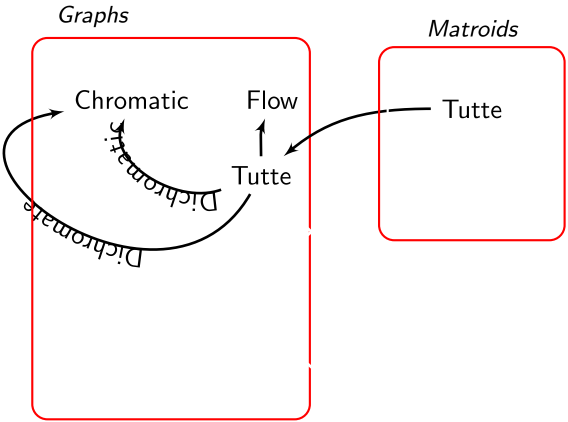
Overview



Tutte polynomial: Specializes to the flow polynomial

$$T_\Gamma(0, 1-n) = (-1)^{|E|-|V|+k(\Gamma)} \phi_\Gamma(n)$$

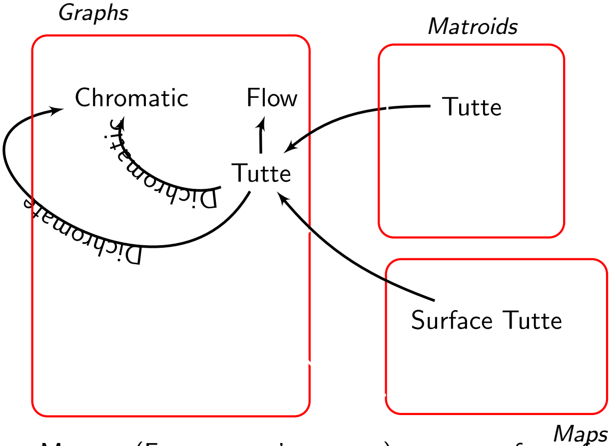
Overview



Tutte polynomial on matroids: if matroid (of rank r) is graphic, Tutte polynomial of graph with same cycle space.

$$T_M(X, Y) = \sum_{A \subset E} (X-1)^{r(E)-r(A)} (Y-1)^{|A|-r(A)}$$

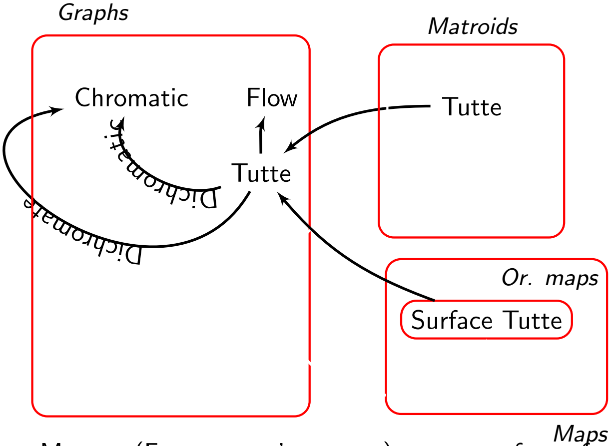
Overview



Tutte on maps: Map $\leftrightarrow$ (Γ, vertex rot'n system); genus g, faces f.

$$\mathcal{T}(M;\mathbf{x},\mathbf{y}) = \sum_{A\subseteq E} x^{|A^c|-f(M)+k(M/A)} y^{|A|-|V|+k(M\backslash A^c)} \prod_{\substack{\text{c.c. } M_i \\ \text{of } M/A}} x_{g(M_i)} \prod_{\substack{\text{c.c. } M_j \\ \text{of } M\backslash A^c}} y_{g(M_j)}$$

Overview



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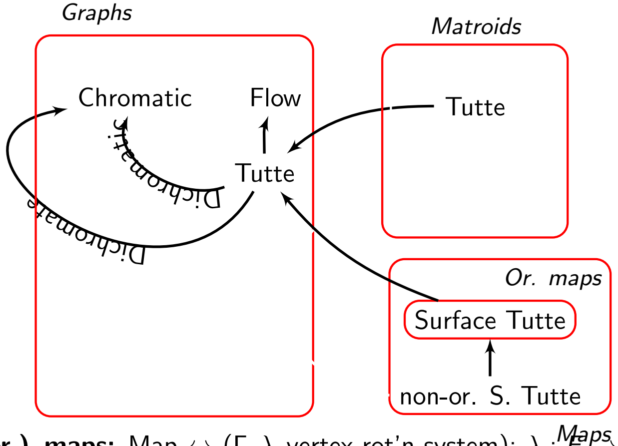
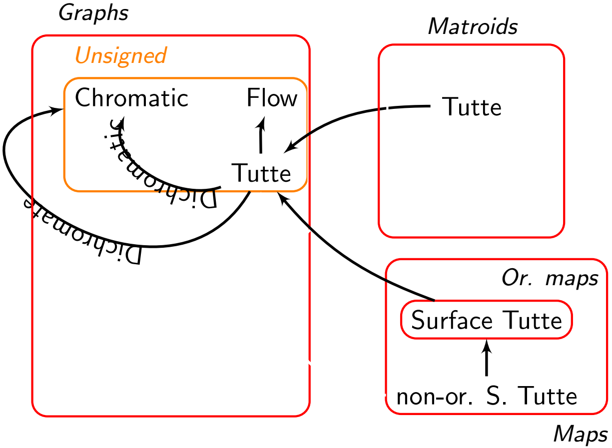


Tutte (non-or.) maps: Map $\leftrightarrow$ $(\Gamma, \lambda, \text{vertex rot'n system})$; $\lambda : E \to \pm 1$ signature. $\bar{g}$ signed genus.

$$\mathcal{T}(M; \mathbf{x}, \mathbf{y}) = \sum_{A \subseteq E} x^{|A^c| - f(M) + k(M/A)} y^{|A| - |V| + k(M \setminus A^c)} \prod_{\substack{\text{c.c. } M_i \\ \text{of } M/A}} x_{\bar{g}(M_i)} \prod_{\substack{\text{c.c. } M_j \\ \text{of } M \setminus A^c}} y_{\bar{g}(M_j)}$$

Overview



Tutte (non-or.) maps: Map $\leftrightarrow$ $(\Gamma, \lambda, \text{rot'n system})$; $\lambda : E \to \pm 1$ signature.

$$\mathcal{T}(M; \mathbf{x}, \mathbf{y}) = \sum_{A \subseteq E} x^{|A^c| - f(M) + k(M/A)} y^{|A| - |V| + k(M \backslash A^c)} \prod_{\substack{\text{c.c. } M_i \\ \text{of } M/A}} x_{\bar{g}(M_i)} \prod_{\substack{\text{c.c. } M_j \\ \text{of } M \backslash A^c}} y_{\bar{g}(M_j)}$$

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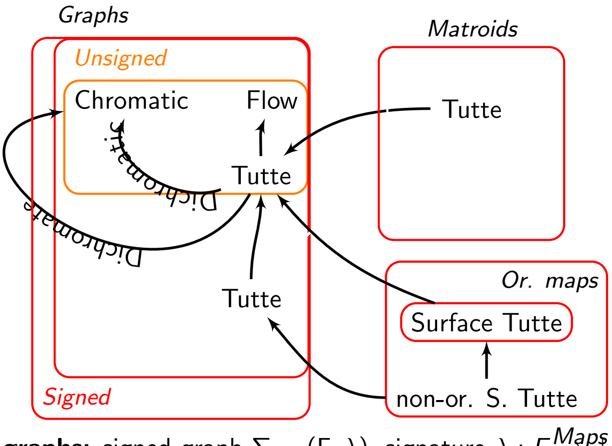
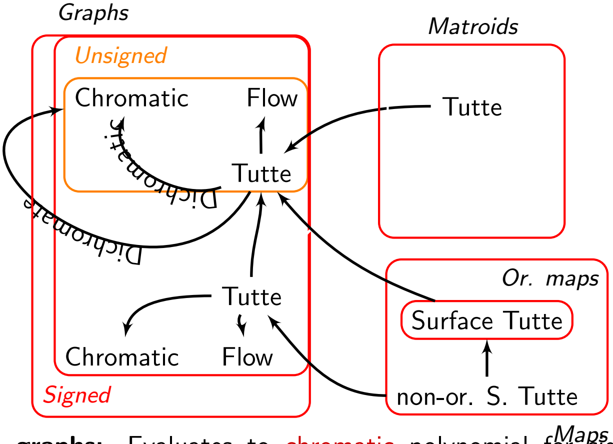


Tutte signed graphs: signed graph $\Sigma = (\Gamma, \lambda)$, signature $\lambda : E \to \pm 1$.

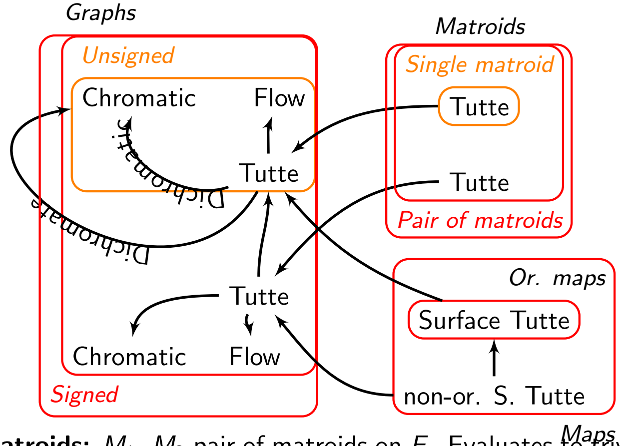
$$T_{\Sigma}(X, Y, Z) = \sum_{A \subseteq E} (X-1)^{k(\Sigma \backslash A^c) - k(\Sigma)} (Y-1)^{|A| - |V| + k_b(\Sigma \backslash A^c)} (Z-1)^{k_u(\Sigma \backslash A^c)}$$

Overview



Tutte signed graphs: Evaluates to chromatic polynomial for signed graphs and flow polynomial for signed graphs.

Overview



Tutte pair matroids: M_1, M_2 pair of matroids on E. Evaluates to trivariate Tutte of $\Sigma = (\Gamma, \lambda)$ for (cycle matroid of Γ, frame matroid of Σ)

$$S_{M_1,M_2}(X,Y,Z) = \sum_{A \subseteq E} (X-1)^{r_1(E)-r_1(A)}(Y-1)^{|A|-r_2(A)}(Z-1)^{r_2(A)+r_1(E)-r_1(A)}.$$

The Tutte polynomial

Graph $\Gamma = (V, E)$.

$$T(\Gamma; x, y) = \sum_{A \subseteq E} (x-1)^{r(\Gamma) - r(\Gamma \setminus A^c)} (y-1)^{n(\Gamma \setminus A^c)},$$

where

- $A^c = E \setminus A$ is the complement of $A \subseteq E$.
- $r(\Gamma) = v(\Gamma) - k(\Gamma)$, $n(\Gamma) = e(\Gamma) - v(\Gamma) + k(\Gamma)$

Properties of the Tutte polynomial

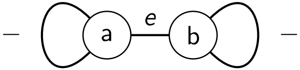
- **Deletion-contraction:**

$$T(\Gamma; x, y) = \begin{cases} T(\Gamma \setminus e; x, y) + T(\Gamma / e; x, y) & \text{for } e \text{ non-bridge edge} \\ yT(\Gamma \setminus e; x, y) & e \text{ a loop} \\ xT(\Gamma / e; x, y) & e \text{ a bridge.} \\ 1 & \Gamma \text{ is the empty graph} \end{cases}$$

- **Universality for deletion-contraction invariants**
- **Duality plane graphs:** $T(\Gamma; x, y) = T(\Gamma^*; y, x)$
- **Chromatic polynomial (counting $\neq 0$ $\mathbb{Z}_n$-tensions):** $(-1)^{r(\Gamma)} n^{k(\Gamma)} T(\Gamma; 1-n, 0)$; $((-1)^{r(\Gamma)} T(\Gamma; 1-n, 0))$
- **Flow polynomial (counting $\neq 0$ $\mathbb{Z}_n$-flows):** $(-1)^{n(\Gamma)} T(\Gamma; 0, 1-n)$.
- and more...

Signed graphs

- A *signed graph* is a pair (Γ, λ), with
  $\Gamma = (V, E)$ a graph (multiple edges and loops allowed),
  $\lambda : E \to \pm 1$ signature map.
- A cycle $(v_1, e_1, v_2, e_2, \ldots, v_1)$ is *balanced* if traverses an even number of negative edges.
  Unbalanced otherwise.
- Connected component is *balanced* if all the cycles are balanced.
  Unbalanced if it contains one unbalanced cycle.



(e is a circuit path edge)

Tutte for signed graphs, pair of matroids

Definition (Trivariate Tutte polynomial of a signed graph)

Signed graph $\Sigma = (\Gamma, \lambda)$

$$T_\Sigma(X,Y,Z) = \sum_{A \subseteq E} (X-1)^{k(\Sigma \setminus A^c) - k(\Sigma)} (Y-1)^{|A| - |V| + k_b(\Sigma \setminus A^c)} (Z-1)^{k_u(\Sigma \setminus A^c)}$$

Definition (Trivariate Tutte polynomial of a pair of matroids)

Matroids $M_1 = (E, r_1)$ and $M_2 = (E, r_2)$ on a common ground set E,

$$S_{(M_1,M_2)}(X,Y,Z) := \sum_{A \subseteq E} (X-1)^{r_1(E) - r_1(A)} (Y-1)^{|A| - r_2(A)} (Z-1)^{r_2(A) + r_1(E) - r_1(A)}$$

$$T_\Sigma(X,Y,Z) = (Z-1)^{-r_M(E)} S_{(M,F)}(X,Y,Z)$$

where M is underlying cycle matroid and F underlying frame matroid of Σ

Properties Tutte for signed graphs I

Deletion-Contraction

$\Sigma = (\Gamma, \lambda)$. e positive edge.

$$T_\Sigma = \begin{cases} T_{\Sigma/e} + T_{\Sigma\setminus e} & e \text{ ordinary in } \Gamma \\ T_{\Sigma/e} + (X-1)\,T_{\Sigma\setminus e} & e \text{ bridge of } \Gamma\text{, circuit path edge of } \Sigma, \\ XT_{\Sigma/e} & e \text{ bridge of } \Gamma\text{, not circuit path edge of } \Sigma, \\ YT_{\Sigma\setminus e} & e \text{ loop of } \Gamma\text{, positive in } \Sigma, \\ 1 + (Z-1)\big[1 + \cdots + Y^{\ell-1}\big] & \Sigma \text{ one vertex with } \ell \geq 1 \text{ negative loops} \\ 1 & \Sigma \text{ has no edges} \end{cases}$$

Properties Tutte for signed graphs II

R be an invariant of signed graphs invariant under switching and multiplicative over disjoint unions. Suppose exists $\alpha, \beta, \gamma, x, y$ and z, with $\gamma \neq 0$, such that, for a signed graph $\Sigma = (\Gamma, \lambda)$ and positive edge $e \in E$,

$$R_\Sigma = \begin{cases} \alpha\, R_{\Sigma/e} + \beta R_{\Sigma\backslash e} & e \text{ ordinary in } \Gamma \text{ and in } \Sigma, \\ \alpha\, R_{\Sigma/e} + \gamma R_{\Sigma\backslash e} & e \text{ ordinary in } \Gamma \text{ and } k_u(\Sigma\backslash e) < k_u(\Sigma), \\ \alpha R_{\Sigma/e} + \frac{\beta(x-\alpha)}{\gamma} R_{\Sigma\backslash e} & e \text{ bridge in } \Gamma, \text{ circuit path edge in } \Sigma, \\ x R_{\Sigma/e} & e \text{ bridge in } \Gamma, \text{ not circuit path edge in } \Sigma, \\ y R_{\Sigma\backslash e} & e \text{ loop in } \Gamma \text{ and in } \Sigma, \\ \beta^{\ell-1}\gamma + (z-\gamma)\sum_{i=0}^{\ell-1} y^{\ell-1-i}\beta^i & \Sigma \text{ one vertex with } \ell \geq 1 \text{ negative loops} \\ 1 & \Sigma \text{ single vertex and no edges} \end{cases}$$

Then, R_Σ is the polynomial in $\alpha, \beta, \gamma, x, y$ and z over $\mathbb{Z}[\gamma, \gamma^{-1}]$

$$R_\Sigma = \alpha^{r_M(E)}\beta^{|E|-r_F(E)}\gamma^{r_F(E)-r_M(E)} T_\Sigma\left(\frac{x}{\alpha}, \frac{y}{\beta}, \frac{z}{\gamma}\right)$$

Definition (G-coloring)

G finite abelian group. A *proper G-coloring* of a signed graph Σ with vertices V is a map $f : V \to G$ such that, for an edge $e = uv$, we have $f(u) \neq f(v)$ if e is positive and $-f(u) \neq f(v)$ if e is negative.

$$P_\Sigma(G) = \begin{cases} -P_{\Sigma/e} + P_{\Sigma\setminus e} & e \text{ ordinary in } \Gamma \text{ and in } \Sigma, \\ -P_{\Sigma/e} + P_{\Sigma\setminus e} & e \text{ ordinary in } \Gamma \text{ and } k_u(\Sigma\setminus e) < k_u(\Sigma), \\ -P_{\Sigma/e} + |G|P_{\Sigma\setminus e} & e \text{ bridge in } \Gamma\text{, circuit path edge in } \Sigma, \\ (|G|-1)P_{\Sigma/e} & e \text{ bridge in } \Gamma\text{, not circuit path edge in } \Sigma, \\ 0P_{\Sigma\setminus e} & e \text{ loop in } \Gamma \text{ and in } \Sigma, \\ |G| - \frac{|G|}{|2G|} & \Sigma \text{ one vertex with } \ell \geq 1 \text{ negative loops} \\ |G| & \Sigma \text{ single vertex and no edges} \end{cases}$$

Hence,

$$P_\Sigma(G) = (-1)^{|V|-k(\Sigma)}|G|^{k(\Sigma)}\,T_\Sigma\left(1-|G|, 0, 1-\frac{1}{|2G|}\right)$$

Definition (Flow)

G finite abelian group. $f : E \to G$ is a *G-flow of a bidirected graph* $(\Gamma = (V, E), \omega)$ if Kirchhoff law is satisfied at each vertex:

$$\sum_{\substack{\text{half edges } (v,e) \\ v \in e}} \omega(v, e) f(e) = 0, \text{ for each } v \in V$$

f is a *G-flow of a signed graph* $\Sigma = (\Gamma, \lambda)$ if bidirection ω is compatible with λ (positive edges: $\to\to/\leftarrow\leftarrow$, negative edges: $\to\leftarrow/\leftarrow\to$).

\# nowhere-zero G-flows on Σ (e positive edge) [DeVos, Rollová, Šámal 13]:

$$q_\Sigma(G) = \begin{cases} q_{\Sigma/e}(G) - q_{\Sigma\setminus e}(G) & \text{if } e \text{ is not a loop of } \Gamma, \\ (|G| - 1) q_{\Sigma\setminus e}(G) & \text{if } e \text{ is a loop of } \Gamma \text{ positive in } \Sigma. \end{cases}$$

and if Σ is a bouquet of ℓ negative loops,

$$q_\Sigma(G) = \frac{1}{|G|} \left[\frac{|G|}{|2G|} (|G| - 1)^\ell + (-1)^\ell (|G| - \frac{|G|}{|2G|}) \right].$$

Hence,

$$q_\Sigma(G) = (-1)^{|E|-|V|+k(\Gamma)} T_\Sigma \left(0, 1 - |G|, 1 - \frac{|G|}{|2G|} \right).$$

Tensions, potential differences I

Definition

G finite abelian group. $\Sigma = (\Gamma, \lambda)$ signed graph, ω compatible bidirection.
$f : E \to G$ is a G-*tension* of Σ with respect to the orientation ω if and only if, for each positive closed walk $W = (v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_1)$,

$$\sum_{i=1}^{k} \left(\omega(v_i, e_i) \prod_{j=1}^{i-1} \lambda(e_j) \right) f(e_i) = 0.$$

f is a G-*potential difference* if and only if f is a G-tension such that, for every walk $W = (v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_1)$ around an unbalanced cycle,

$$\sum_{i=1}^{k} f(e_i) \in 2G.$$

Note: $\frac{|G|^{k(\Sigma)}}{|2G|^{k_u(\Sigma)}}$ proper G-colorings $\leftrightarrow$ 1 nowhere-zero G-potential difference.
For each walk around an unbalanced cycle,

$$\sum_{i=1}^{k} f(e_i) \in u + 2G.$$

for some $u \in G$ depending on the connected component.

Tensions, potential differences II

For positive edge e, and $u \in G$, the number of nowhere-zero G-tensions where

$$\sum_{i=1}^{k} f(e_i) \in u + 2G.$$

for each unbalanced cycle in each connected component we have satisfies

$$p_\Sigma(G;u) = \begin{cases} p_{\Sigma\backslash e}(G;u) - p_{\Sigma/e}(G;u) & \text{if } e \text{ is ordinary in } \Gamma \text{ and in } \Sigma, \\ |2G|p_{\Sigma\backslash e}(G;u) - p_{\Sigma/e}(G;u) & e \text{ ordinary in } \Gamma \text{ and } k_u(\Sigma\backslash e) < k_u(\Sigma), \\ \frac{|G|}{|2G|}p_{\Sigma\backslash e}(G;u) - p_{\Sigma/e}(G;u) & e \text{ bridge in } \Gamma, \text{ circuit path edge in } \Sigma, \\ (|G|-1)p_{\Sigma\backslash e}(G;u) & e \text{ bridge in } \Gamma, \text{ not circuit path edge in } \Sigma, \\ 0 & e \text{ loop in } \Gamma \text{ and in } \Sigma \text{ (positive loop)}, \end{cases}$$

and $p_\Sigma(G;u) = \left\{ \begin{matrix} |2G|-1 & u \in 2G \\ |2G| & u \notin 2G \end{matrix} \right\}$ for a vertex with $\ell \geq 1$ negative loops.

Hence, the number of nowhere-zero G-potential differences:

$$p_\Sigma(G) = (-1)^{r(\Gamma)}|2G|^{k_u(\Sigma)}T_\Sigma\left(1-|G|, 0, 1-\frac{1}{|2G|}\right),$$

And, if G unbalanced and connected, the number of nowhere-zero G-tensions

$$t_\Sigma(G) = (-1)^{r(\Gamma)}|2G|\left[T_\Sigma\left(1-|G|, 0, 1-\frac{1}{|2G|}\right) + \left(\frac{|G|}{|2G|}-1\right)T_\Sigma(1-|G|,0,1)\right].$$

Relation of Tutte for signed graphs to other polynomials

- Includes Zaslavsky's dichromatic polynomial for signed graphs/ partition function for mixed Potts model on a signed graph (g.f. for states by no. of improperly coloured edges)
- Tutte polynomial for pairs of matroids is Welsh–Kayibi's *linking polynomial* (frame matroid and cycle matroid relevant pair of matroids for signed graphs)
- Fits in the unifying framework of "canonical Tutte polynomials" of Krajewski, Moffat and Tanasa '18 of Hopf algebras and Tutte polynomials, and the extension by Dupont, Fink and Moci '18.
- Evaluation of the surface Tutte polynomial introduced by G., Litjens, Regts, Vena +'20 (generalization of G., Krajewski, Regts, Vena '18), which is *not* the canonical Tutte polynomial for maps but rather akin to Tutte's universal V-function of graphs

Loose ends

- Other evaluations of the trivariate Tutte polynomial for signed graphs with combinatorial interpretations (not evaluations of the Tutte polynomial of the underlying cycle matroid or frame matroid)?
- Enumerating flows and tensions for gain graphs, for biased graphs more generally: gives the canonical Tutte polynomial?
- Hochstättler and Wiehe '21 have constructed a trivariate Tutte polynomial for digraphs that enumerates Neumann Lara flows and acyclic colourings that is not the canonical Tutte polynomial. When is the "dichromate" (enumerating the analogue of nowhere-zero tensions and flows) equal to the canonical Tutte polynomial?

Thank you for your attention!