GCD RESULTS ON SEMIABELIAN VARIETIES AND A CONJECTURE OF SILVERMAN

AMOS TURCHET - Roma Tre University

Joint with F. Barroero & L. Capuano

GEOMETRIC DIVISIBILITY SEQUENCE [Silverman] $k = \overline{k}$

$G \to C$ group scheme over non-sing. proj. curve $C/k$

$G/\kappa(C)$ generic fiber of $G$

Assume $G$ is IRREDUCIBLE, COMMUTATIVE, no UNIPOTENT PART

$\mathbb{Z}_p : C \to G$ section corresponding to $P \in G$

$\mathbb{G} : C \to G$ identity section

$D_p$ divisor on $C$ associated to $\mathbb{Z}_p \times \mathbb{G}$

A geometric divisibility sequence is a seq. of divisor $(D_{np})_{n \geq 1}$

$L$, $n | m \implies D_{np} < D_{nq}$
**Examples**

1. \( g : \mathbb{G}_m \to \mathbb{P}^1 \)
   \[ P = \frac{a(t)}{b(t)} \in G = \mathbb{G}_m(\mathbb{k}(t)) = \mathbb{k}(t)^x \]

   \[ \text{supp } D_p = \{ t : a(t) = b(t) \} \]
   \[ \text{supp } D_{np} = \{ t : a^*(t) = b^*(t) \} \]

   \[ m \ln \Rightarrow D_{mp} \leq D_{np} \quad \text{in general } \text{supp } D_{np} \text{ not bounded} \]

2. \( g : E \to \mathbb{P}^1 \) elliptic scheme over \( \mathbb{P}^1 \)

   \[ P \in G = E_{/\mathbb{k}(t)} \mapsto (x_P, y_P) \quad x_P, y_P \in k(t) \]
   \[ x_P = \frac{A_p}{D_p^2}, \quad A_p, D_p \in k[t] \]

   \[ D_p = \frac{1}{2} \text{div}(x_P)_\infty = \text{div}(D_p)_\infty \]

   \[ m \ln \Rightarrow D_{mp} \| D_{np} \text{ as polynomials} \]

   **OSS:** \( P \) not torsion \( \Rightarrow \) \text{supp } D_{np} \text{ not bounded}
SILVERMAN CONJECTURE (function field version)

Assume

a) $G$ has dimension $\dim_k G \geq 2$

b) The subgroup generated by $P$ is $\mathbb{Z}$-dense in $G$

Then:

$D_{np} = D_p$ for infinitely many $n$

MOTIVATION: result of Bugeaud-Covaya-Zenner in $\mathbb{F}_m^2 / \mathbb{F}_2$

- $G = \mathbb{G}_m^2$ Ailon-Rudnick '04 $\Rightarrow \gcd(a^n-1, b^n-1) | c$
  - Ostafe '16 general case Indep. of $n$

- $G = E_1 \times E_2$ all curves
  - $E_1, E_2$ isotrivial $\Rightarrow$ Silverman '04
  - general case $\Rightarrow$ Ghioca-Xia-Tucker '18
**THEOREM** (Barroero - Capuano - T.)

Conjecture holds for $G = A \times \mathbb{G}_m$, $A$ ab variety, i.e. for every $P \in G/k(c)$ s.t. $P^2 = G$:

$$D_{np} = D_p$$

for infinitely many $n \geq 1$

Moreover, there exists an effective divisor $D$ on $C$ s.t.

$$D_{np} < D$$

for every $n \geq 1$

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**EXAMPLE**

$G = E \times \mathbb{G}_m \to C = \mathbb{P}^1$

$P = (Q, t) \in E \times \mathbb{G}_m / k(c) = G$

$$D_p = \gcd(D_Q, D_t)$$

There exists $D \in \text{Div}(C)$ s.t. $D_{np} = \gcd(D_{na}, D_{np}) \# D$ for every $n \geq 1$