Isospectral magnetic graphs

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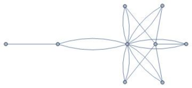
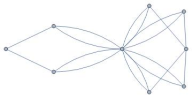
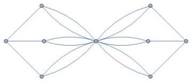
Overview:

1. Motivation
2. Graphs and magnetic Laplacians
3. Spectral preorder of magnetic graphs
4. Construction of isospectral magnetic graphs
5. Outlook

(Joint work with John Fabila-Carrasco (U. Edinburgh, UK) and Olaf Post (U. Trier, Germany))

1. Motivation: What do we want to explain?

- What is the geometrical reason behind the fact that the graphs below are isospectral for the **discrete magnetic Laplacian** with standard weights?

  - For **any magnetic flux** through the graph (including zero flux)!
  - In this talk we use only **standard weights**:

$$w(v) = \deg(v) \ , \ v \in V \quad \text{and} \quad w_e = 1 \ , \ e \in E \ .$$

  Everything for more general weights (normalised).

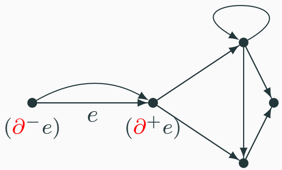
- Why standard weights?
  - Much more difficult to obtain: e.g., graphs with 9 vertices (e.g., Bulter-Grout '11)
    - combinatorial weights: 15% isospectral; standard weights: 0.4% isospectral

2. Graphs and discrete magnetic Laplacians

We are considering graphs with magnetic potential on it.

- **Oriented multigraphs:**

  $G = (V, E, \partial)$ with $\partial\colon E \to V \times V$,
  $\partial e = (\partial^- e, \partial^+ e)$.



- **Magnetic vector potential on edges:** $\alpha\colon E \to \mathbb{R}/\mathbb{Z} \cong [0, 2\pi)$.

  Since the magnetic field satisfies: $\mathbf{B} = d\mathbf{A}$ we define

  - **Gauge equiv.:** $\alpha \sim \alpha'$ if there is $\xi\colon V \to \mathbb{R}/\mathbb{Z}$ with $\alpha - \alpha' = d\xi$, where $(d\xi)_e = \xi(\partial^+ e) - \xi(\partial^- e)$.

Hilbert spaces associated to weighted graph:

Let G a graph graph (with standard weights):

$$\ell_2(V) := \{f\colon V \to \mathbb{C}\} \quad (0-\text{forms}) \quad \text{and} \quad \ell_2(E) := \{\eta\colon E \to \mathbb{C}\} \quad (1-\text{forms})\,.$$

Inner product:

$$\langle f, g\rangle_{\ell_2(V)} = \sum_{v\in V} f(v)\overline{g(v)}\deg(v) \quad \text{and} \quad \langle \eta, \zeta\rangle_{\ell_2(E)} = \sum_{e\in E} \eta_e \overline{\zeta_e}\,.$$

Definition

Let $\mathbf{G} = (G, \alpha)$ be a graph with magnetic potential $\alpha\colon E(\mathbf{G}) \to \mathbf{I} = \mathbb{R}/2\pi\mathbb{Z}$

- The **twisted derivative** is $d_\alpha\colon \ell_2(V) \to \ell_2(E)$ with

$$(d_\alpha f)_e = e^{i\alpha_e/2} f(\partial^+ e) - e^{-i\alpha_e/2} f(\partial^- e)\,.$$

Discrete magnetic Laplacians (a geometric approach):

Definition

Let **G** $= (G, \alpha)$ a weighted graph with vector potential α. The **Discrete magnetic Laplacian (DML)** is:

$$\Delta_\alpha : \ell_2(V) \to \ell_2(V) \text{ given by } \Delta_\alpha = d_\alpha^* d_\alpha$$

$$(\Delta_\alpha f)(v) = f(v) - \frac{1}{\deg(v)} \sum_{e \in E_v} e^{i \overset{\frown}{\alpha_e}(v)} f(v_e) \, .$$

- Oriented evaluation: $\overset{\frown}{\alpha_e}(v) = \begin{cases} -\alpha_e, & \text{if } v = \partial^+(e) \\ \alpha_e, & \text{if } v = \partial^-(e). \end{cases}$
- $v_e \equiv$ vertex opposite to v along e and $E_v \equiv$ edges “touching” the vertex v.

Signless standard Laplacian if $\alpha = \pi$: $(\Delta_\pi f)(v) = f(v) + \frac{1}{\deg(v)} \sum_{e \in E_v} f(v_e)$.

Facts in relation to the magnetic potential:

- If $\alpha \sim \alpha' \Rightarrow \Delta_\alpha \cong \Delta_{\alpha'} \Rightarrow \sigma(\Delta_\alpha) = \sigma(\Delta_{\alpha'})$.
- If G is a tree $\Rightarrow$ any $\alpha \sim 0$ and $\Delta_\alpha \cong \Delta_0$ and $\sigma(\Delta_\alpha) = \sigma(\Delta_0)$.

3. Spectral ordering for finite graphs

Let $\mathbf{G} = (G, \alpha)$ be a finite magnetic graph of order $|V| = n$.
Denote the **spectrum** of Δ_α by

$$\sigma(\mathbf{G}) = \sigma(\Delta_\alpha) := \{\lambda_1(\mathbf{G}) \leq \cdots \leq \lambda_n(\mathbf{G})\} \; .$$

Definition

Let $\mathbf{G} = (G, \alpha)$, $\widetilde{\mathbf{G}} = (\widetilde{G}, \widetilde{\alpha})$ be magnetic graphs with orders n resp. $\widetilde{n}$.
$\mathbf{G}$ is **spectrally smaller than $\widetilde{\mathbf{G}}$ with shift $q \in \mathbb{N}_0$**

Notation : $\mathbf{G} \overset{q}{\preccurlyeq} \widetilde{\mathbf{G}}$ if $n \geq \widetilde{n}$ and $\lambda_k(\mathbf{G}) \leq \lambda_{k+q}(\widetilde{\mathbf{G}})\,,\; k = 1, \ldots, n-q\;.$

Remarks:

- $\mathbf{G} \preccurlyeq \widetilde{\mathbf{G}} \overset{1}{\preccurlyeq} \mathbf{G}$ just means that the corresponding eigenvalues **interlace**.
- The preorder $\preccurlyeq$ describes the spectral effect of a graph perturbation.

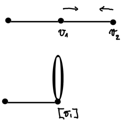
Two important elementary perturbations magnetic graphs

1) **Vertex contraction**

Proposition (Fabila-Carrasco, Ll., Post, ’20)

Let $\mathbf{G} = (G, \alpha)$ *and* $V_0 \subset V(G)$ *and denote the graph* $\widetilde{\mathbf{G}} = \mathbf{G}/\!\sim_{V_0}$ *the graph with the* V_0 *vertices contracted. Then*

$$\mathbf{G} \preccurlyeq \widetilde{\mathbf{G}} \overset{q}{\preccurlyeq} \mathbf{G}\,, \quad \text{where} \quad q = |G| - |\widetilde{G}| \quad \text{is the shrinking number.}$$



2) Vertex virtualisation

Let **G** $= (G, \alpha)$ and $V_0 \subset V(G)$.

- Consider the magnetic Laplacian restricted to functions with Dirichlet conditions on V_0 or, equivalently,
- The virtualised Laplacian is the compression of Δ_α to the subspace $\ell_2(V \setminus V_0)$:

$$\iota \colon \ell_2(V \setminus V_0) \to \ell_2(V) \quad \text{and} \quad \Delta^+_{V_0} = \iota^* \Delta_\alpha \iota \,.$$

Notation for the virtualised spectra: $\sigma(\mathbf{G}^+_{V_0}) := \sigma(\Delta^+_{V_0})$

Proposition (Fabila-Carrasco, Ll., Post, '18)

Let **G** $= (G, \alpha)$ *and* $V_0 \subset V(G)$ *with* $q = |V_0|$ *and denote the* V_0*-virtualised graph by* $\mathbf{G}^+_{V_0}$. *Then*

$$\mathbf{G} \preccurlyeq \mathbf{G}^+_{V_0} \overset{q}{\preccurlyeq} \mathbf{G} \,, \quad \textit{where} \quad q = |V_0| \,.$$

- For a systematic study the spectral order relation under perturbations of the graph and many applications

⇝ see Fabila-Carrasco, Ll., Post, *Spectral preorder and perturbations of discrete weighted graphs*, Math. Ann. 2020.

4. Construction of isospectral graphs - in three steps

Step 1: Building block.

Let $\mathbf{G} = (G, \alpha)$ **any** finite magnetic graph with spectrum $\sigma(\mathbf{G})$.

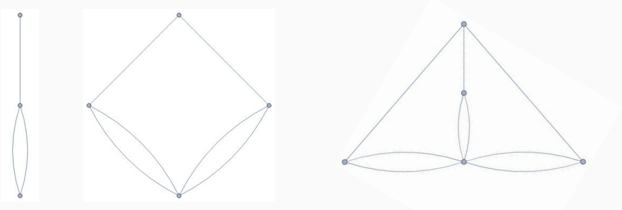
Graph G

Step 2: Isospectral frame. Let $\mathbf{G} = (G, \alpha)$ be a building block.

Choose $V_0 \subset V(G)$ and for any $p \in \mathbb{N}$ define

$$F_p = F_p(\mathbf{G}, V_0) = \left(\bigsqcup^{p} G \right) / \sim_{V_0} \quad \text{with} \quad F_1 = G \, .$$

Example: Let $\mathbf{G}$ be the building block and $V_0 = \{v_0, v_1\}$ (upper/lower vertices of G).



$F_1 = G$ $\qquad$ F_2 $\qquad$ F_3 $\qquad \ldots$

Theorem (Fabila-Carrasco, Ll., Post, '21)

The spectrum of the frames is: $\sigma\left(F_p(\mathbf{G}, V_0)\right) = \sigma(\mathbf{G}) \uplus \sigma(\mathbf{G}^{+}_{V_0})^{p-1}, \, p \in \mathbb{N}.$

Step 3: Isospectral graphs. Let $\mathbf{G} = (G, \alpha)$ be a magnetic building block.

- Choose $V_0 \subset V(G) \rightsquigarrow$ isospectral frames $\{F_p(G, V_0)\}_{p \in \mathbb{N}}$ (steps 1 and 2).

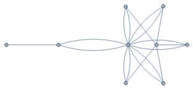
How can we glue the frames?

- Choose a **distinguished vertex** $v_1 \in V_0$
- Choose an s **partition of the natural number** r:
  $A = (a_1, \ldots, a_s)$ with $r = a_1 + \cdots + a_s$.

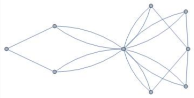
Contract (glue) the frames $F_{a_1}, \ldots, F_{a_s}$ at the distinguished vertex v_1

$$F(A) = \left(\bigsqcup_{a \in A} F_a \right) \Big/ \sim_{v_1} \; .$$

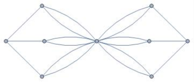
Example: Consider the three partitions of $r = 6$ of length $s = 2$.



$A = (1, 5)$



$A = (2, 4)$



$A = (3, 3)$

Theorem (Fabila-Carrasco, Ll., Post, '21)

Let $\boldsymbol{G} = (G, \alpha)$ be a magnetic building block

- *For any $V_0 \subset V(G) \rightsquigarrow$ isospectral frames $\{F_p(G, V_0) \mid p \in \mathbb{N}\}$.*
- *For any $v_1 \in V_0$ distinguished vertex to "glue" the frames.*
- *Consider A, B two different s partitions of $r \geq 4$ i.e., $A = (a_1, \dots, a_s)$, $B = (b_1, \dots, b_s)$ with*

$$r = a_1 + \cdots + a_s = b_1 + \cdots + b_s \ .$$

Then

1) *$F(A)$ and $F(B)$ are isospectral and $F(A) \ncong F(B)$.*
2) *$\sigma\left(F(A)\right) = \sigma\left(F(B)\right) = \sigma(\boldsymbol{G}) \uplus \sigma(\boldsymbol{G}^{+}_{V_0})^{(r-s)} \uplus \sigma(\boldsymbol{G}^{+}_{v_1})^{(s-1)}$.*

Remark:

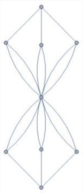
- Note the $\sigma(F(A))$ **only** depends on r and the length s of the partition and not on the particular decomposition.
- If $P(r, s)$ is the set of partitions of r of length $s \rightsquigarrow$ one produces **$|P(r, s)|$-families non isomorphic isospectral graphs** for Δ_α.
- Generlises Butler-Grout's '11 examples where $G = P_3$ and frames where diamond graphs (and $\alpha = 0$).

Idea of the proof

- The proof isospectral property is based the following spectral relations

1) $F(A)^+_{v_1} \overset{1}{\preccurlyeq} F(A) \preccurlyeq F(A)^+_{v_1}$.

2) $\Big(\bigsqcup\limits_{a\in A} F_a\Big) \preccurlyeq F(A) \overset{s-1}{\preccurlyeq} \Big(\bigsqcup\limits_{a\in A} F_a\Big)$.



$\rightsquigarrow$ the spectra of $F(A)^+_{v_1}$ and $\Big(\bigsqcup\limits_{a\in A} F_a\Big)$ are known explicitly;

$\rightsquigarrow$ exploit that frames have **high symmetry/multiplicity**.

$\rightsquigarrow$ high multiplicity of eigenvalues of the spectrum of the sandwiching graphs $F(A)^+_{v_1}$ and $\Big(\bigsqcup\limits_{a\in A} F_a\Big)$.

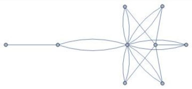
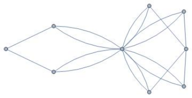
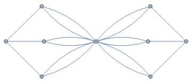
- $F(A) \not\cong F(B)$ because in the corresponding **degree lists** the partitions appear explicitly

$$(a_1, \dots, a_s, \dots) \neq (b_1, \dots, b_s, \dots)$$

Outlook

What have we done?

- Explained what is the geometrical reason explaining the magnetic isospectrality of e.g.,

- Control the spectral spreading of eigenvalues under elementary perturbations of the graph like **vertex contraction** and **virtualisation**.

$$\mathbf{G} \preccurlyeq \widetilde{\mathbf{G}} \overset{q}{\preccurlyeq} \mathbf{G}\,.$$

Results from: Fabila-Carrasco, Ll., Post

- *A geometric construction of isospectral magnetic graphs*, preprint 2021; arXiv:math.CO.???
- *Spectral preorder and perturbations of discrete weighted graphs*, Mathematische Annalen 2020 (49pp.)
- *Spectral gaps and discrete magnetic Laplacians*, Linear Algebra and its Applications **547** (2018) 183-216.