Isometries of Wasserstein spaces

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What is a Wasserstein space?

- roughly speaking: the space of sufficiently concentrated probability measures endowed with a metric which is calculated by means of *optimal transport* (Monge: 1781, Kantorovich: WW II, Villani: 2010, Figalli: 2018)
- more precisely: for a Polish space (X, ρ) and a parameter $0 < p < \infty$ the p-*Wasserstein space* is

$$\mathcal{W}_p(X) = \left\{ \mu \in \mathcal{P}(X) \,\middle|\, \int_X \rho(x, \hat{x})^p \, \mathrm{d}\mu(x) < \infty \text{ for some } \hat{x} \in X \right\}$$

endowed with the p-*Wasserstein distance*

$$d_{\mathcal{W}_p}(\mu, \nu) := \left(\inf_{\pi \in \Pi(\mu,\nu)} \int_{X^2} \rho(x, y)^p \, \mathrm{d}\pi(x, y) \right)^{\min\left\{\frac{1}{p}, 1\right\}}.$$

where $\Pi(\mu, \nu) = \left\{ \pi \in \mathcal{P}\left(X^2\right) \,\middle|\, \pi_1 = \mu, \pi_2 = \nu \right\}$ is the collection of all *transport plans* between μ and ν

Probabilistic and dynamical interpretations

- a probabilistic reformulation of the p-Wasserstein distance:

$$d_{\mathcal{W}_p}(\mu,\nu) = \left(\inf_{(\mathrm{x},\mathrm{y}):\, \mathrm{x}\sim\mu,\, \mathrm{y}\sim\nu} \mathrm{E}\left(d(\mathrm{x},\mathrm{y})^p\right) \right)^{\min\left\{\frac{1}{p},1\right\}}.$$

- a dynamical interpretation: Benamou-Brenier formula (fluid mechanics or "Eulerian" formulation):

$$d^2_{\mathcal{W}_2}(\mu,\nu) = \inf_{(\rho,v)\in V(\mu,\nu)} \int_0^1 \int_{R^n} \rho_t(x) \left\| v_t(x) \right\|^2 \mathrm{d}x\mathrm{d}t,$$

where $\{\rho_t\}_{t\in[0,1]}$ is the weak solution of the linear transport equation

$$\frac{\partial \rho_t}{\partial t} + \nabla_x \cdot (\rho_t v_t) = 0$$

with initial and final conditions $\rho_0 = \mu, \rho_1 = \nu$

Motivation and basic notions

- when working in a metric setting, a natural question arises: *how do isometries look like?*
- classical results:
  - the Banach–Lamperti theorem describing all linear non-surjective isometries of L^p spaces
  - the Banach–Stone theorem on the group of all linear isometries of commutative C^*-algebras ($C(K)$)
- Bertrand and Kloeckner: a series of papers on isometries of quadratic Wasserstein spaces over various metric spaces
- Kloeckner:[1] a description of the isometry group of $\mathcal{W}_2(E)$ for $\dim(E) < \infty$
- main question: is $\mathcal{W}_p(X)$ more symmetric than X?
- isometric rigidity: $\mathrm{Isom}(\mathcal{W}_p(X)) = \mathrm{Isom}(X)$

[1] B. Kloeckner, *A geometric study of Wasserstein spaces: Euclidean spaces,* Annali della Scuola Normale Superiore di Pisa - Classe di Scienze **IX** (2010), 297–323.

Basic notions, notation

Definition (Push-forward)

For a measurable map $g\colon X \to X$ the induced *push-forward map* $g_\#\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is defined by

$$\left(g_\#(\mu)\right)(A) = \mu(g^{-1}[A]) \qquad (A \subseteq X \text{ Borel set}, \ \mu \in \mathcal{P}(X))$$

where $g^{-1}[A] = \{x \in X \,|\, g(x) \in A\}$. We call $g_\#(\mu)$ the *push-forward* of μ with g. If $\psi \in \mathrm{Isom}(\mathbb{R})$, then the push-forward map $\psi_\#$ is an isometry of $\mathcal{W}_p(X)$, and the embedding

$$\#\colon\ \mathrm{Isom}(X) \to \mathrm{Isom}\left(\mathcal{W}_p(X)\right), \qquad \psi \mapsto \psi_\#$$

is a group homomorphism. Isometries of the form $\psi_\#$ are called *trivial isometries*.

Basic notions, notation

- p-Wasserstein distance expressed by cumulative distribution and quantile functions:
  - Vallender:[2]

$$d_{\mathcal{W}_1}(\mu,\nu) = \int_{-\infty}^{\infty} |F_\mu(x) - F_\nu(x)| \, \mathrm{d}x = \int_0^1 \left|F_\mu^{-1}(x) - F_\nu^{-1}(x)\right| \, \mathrm{d}x$$

  - this can be generalized:[3]

$$d_{\mathcal{W}_p}(\mu,\nu) = \left(\int_0^1 \left|F_\mu^{-1} - F_\nu^{-1}\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \qquad (p > 1, \ \mu,\nu \in \mathcal{W}_p(\mathbb{R})),$$

[2] S. S. Vallender, *Calculation of the Wasserstein distance between probability distributions on the line,* Theory Probab. Appl. 18 (1973), 784–786.

[3] C. Villani, *Topics in optimal transportation,* Graduate studies in Mathematics vol. 58, American Mathemtical Society, Providence, RI, 2003.

The discrete case

Theorem (Gehér, Titkos, V., *J. Math. Anal. Appl.* **480** (2019), 123435)

For $p \in (0, \infty)$ and $f : \mathcal{W}_p(\mathcal{X}) \to \mathcal{W}_p(\mathcal{X})$ an isometric embedding, there exists a unique family of measures

$$\Phi := \left(\varphi_{x,t}\right)_{x \in \mathcal{X}, t \in (0,1]} \in \mathcal{M}(\mathcal{X})^{\mathcal{X} \times (0,1]}$$

that satisfies the properties

(a) *for all $x \neq y$: $S_{\varphi_{x,1}} \cap S_{\varphi_{y,1}} = \emptyset$,*

(b) *for all $x \in \mathcal{X}$ and $0 < t \leq 1$: $\varphi_{x,t}(\mathcal{X}) = t$,*

(c) *for all $x \in \mathcal{X}$ and $0 < s < t \leq 1$: $\varphi_{x,s} \leq \varphi_{x,t}$,*

and that generates f in the sense that $f(\mu) = \sum_{x \in S_\mu} \varphi_{x, \mu(\{x\})}$.
Conversely, every $\mathcal{X} \times (0, 1]$-indexed family of measures satisfying (a)–(c) *generates an isometric embedding.*

The discrete case

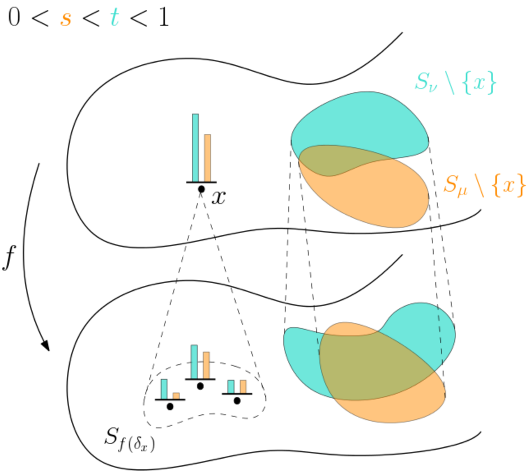


Figure: $f(\mu)|_{S_{f(\delta_x)}} \leq f(\nu)|_{S_{f(\delta_x)}}$.

Unit interval, real line, and Euclidean spaces

Theorem (Gehér, Titkos, V., *Trans. Amer. Math. Soc.* (2020) & G-T-V, arXiv:2102.02037 (2021))

For $0 < p < 1$, *the Wasserstein space* $\mathcal{W}_p(X)$ *is isometrically rigid* **for every Polish space** X. *Moreover, the isometry group of* $\mathcal{W}_p(X)$ *depending on* $X \in \{[0,1], \mathbb{R}, E\}$ *and* $p \geq 1$ *is isomorphic to:*

$\mathrm{Isom}\left(\mathcal{W}_p(X)\right)$	$X = [0,1]$	$X = \mathbb{R}$	$X = E$
$p = 1$	$C_2 \times C_2$	$\mathrm{Isom}(\mathbb{R})$	$\mathrm{Isom}(E)$
$p > 1,\ p \neq 2$	C_2	$\mathrm{Isom}(\mathbb{R})$	$\mathrm{Isom}(E)$
$p = 2$	C_2	$\mathrm{Isom}(\mathbb{R}) \ltimes \mathrm{Isom}(\mathbb{R})$	$\mathrm{Isom}(E) \ltimes O(E)$

E stands for a separable real Hilbert space. Color code: Kloeckner (2010), G-T-V (2020-21), Kloeckner: $\dim(E) < \infty$, G-T-V: $\dim(E) = \infty$.

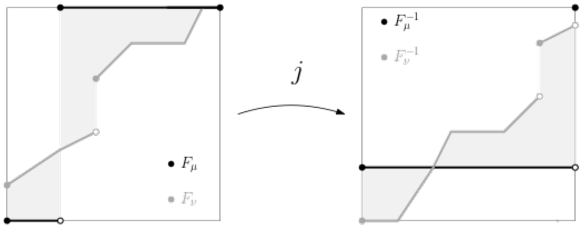
The unit interval

Theorem (G-T-V, TAMS (2020))

Every isometric embedding of $\mathcal{W}_1([0,1])$ *is surjective, and*

$$\mathrm{Isom}(\mathcal{W}_1([0,1])) = \{\mathrm{id}, r_\#, j, r_\# j\} \simeq C_2 \times C_2$$

(Klein group), where r *is the reflection of* $[0,1]$, *and* j *is the flip operation defined by* $F_{j(\mu)} = F_\mu^{-1}$



The action of isometries of $\mathcal{W}_1([0,1])$

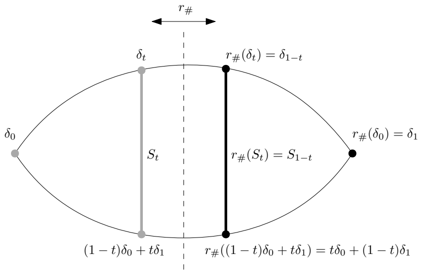
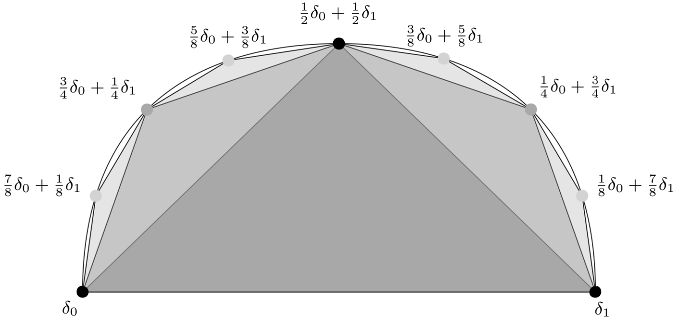


Figure: The action of $r_\#$ and j on $\mathcal{W}_1([0,1])$

The shape of $\mathcal{W}_2([0,1])$



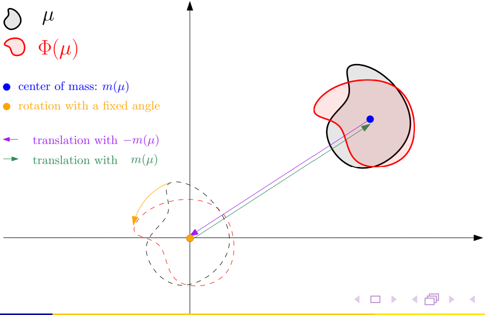
Wasserstein-Hilbert spaces (G-T-V, arXiv:2102.02037)

(a) If $p \neq 2$, then Φ is necessarily a push-forward of an isometry ψ of E, that is, $\Phi(\mu) = \psi_{\#}\mu$.

(b) If $p = 2$ and E is infinite dimensional then

$$\Phi(\mu) = \left(\psi \circ t_{m(\mu)} \circ R \circ t_{m(\mu)}^{-1}\right)_{\#} \mu \qquad (\mu \in \mathcal{W}_2(E)),$$

where $\psi \colon E \to E$ is an affine, $R \colon E \to E$ is a linear isometry, and $t_{m(\mu)} \colon E \to E$ is the translation on E by the barycenter $m(\mu)$ of μ



Wasserstein-Hilbert spaces (G-T-V, arXiv:2102.02037)

- main tool: the Wasserstein potential defined by

$$\mathcal{T}_\mu^p \colon E \to \mathbb{R}, \quad x \mapsto d_{\mathcal{W}_p}^p(\mu, \delta_x) = \int_E \|x-y\|^p \,\mathrm{d}\mu(y)$$

  from which *atoms* can be recovered if $1 \leq p < \infty$, $2 \nmid p$ by

$$\lim_{h\to 0} \frac{\sum_{j=0}^{2k} \binom{2k}{j}(-1)^j \mathcal{T}_\mu^p\left(x+(k-j)h\right)}{\left(\sum_{j=0}^{2k} \binom{2k}{j}(-1)^j |k-j|^p\right)\|h\|^p} = \mu(\{x\})$$

- for $2 \mid p$ the potential *does not contain enough information about the measure*, still, we proved rigidity by different techniques

Future plan

Describe the isometric structure of p-Wasserstein spaces over:

- graphs with the shortest path distance
- Banach spaces, length spaces, non-branching spaces: no inner product structure $\Rightarrow$ genuinely new methods are needed
- spheres, tori: Fourier method combined with the Wasserstein potential technique
- projective spaces: represent the sets of pure states in quantum mechanics

Thank you for your kind attention!