Ergodic theory for energetically open fluid systems

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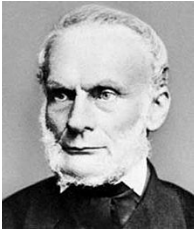
PDEs describing far from equilibrium systems,
8 ECM Portorož, 20 June – 26 June 2021

Motto



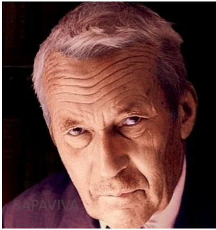
Rudolf Clasius
1822–1888

Basic principles of thermodynamics of closed systems

Die Energie der Welt ist constant. Die Entropie der Welt strebt einem Maximum zu.

Turbulence - ergodic hypothesis

Time averages along trajectories of the flow converge, for large enough times, to an ensemble average given by a certain probability measure



Andrey Nikolaevich Kolmogorov
1903–1987

Navier–Stokes–Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

Newton's Second law (momentum balance)

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S} + \varrho \mathbf{g}$$

Second law of thermodynamics (entropy balance)

$$\partial_t (\varrho s(\varrho, \vartheta)) + \mathrm{div}_x (\varrho s(\varrho, \vartheta) \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \frac{1}{\vartheta} \left(\mathbb{S} : \mathbb{D}_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Newton's rheological law

$$\mathbb{S}(\vartheta, \mathbb{D}_x \mathbf{u}) = \mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta(\vartheta) \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Fourier's law

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta$$

Boundary conditions

Closed systems

$$\textbf{impermeability:} \quad \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0, \ \textbf{no–slip:} \quad \mathbf{u}\times\mathbf{n}|_{\partial\Omega} = 0$$

$$\textbf{thermal insulation:} \quad \mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0$$

Open systems

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_B, \ \textbf{inflow } \Gamma_{\rm in} : \mathbf{u}_B\cdot\mathbf{n} < 0, \ \textbf{outflow } \Gamma_{\rm out} : \mathbf{u}_B\cdot\mathbf{n} > 0$$

$$\varrho|_{\Gamma_{\rm in}} = \varrho_B$$

heat flow: $\varrho e(\varrho,\vartheta)(\mathbf{u}_B\cdot\mathbf{n}) + \mathbf{q}\cdot\mathbf{n} = f_{i,B}(\mathbf{u}_B\cdot\mathbf{n})$ on $\Gamma_{\rm in}$, $\mathbf{q}\cdot\mathbf{n} = 0$ on $\Gamma_{\rm out}$

alternatively

$$\vartheta = \vartheta_B \text{ on } \partial\Omega$$

Necessary ingredients

- **Global existence:** The problem admits global–in–time solutions defined for all $t \geq t_0$ for any admissible data

- **Dissipativity (in the sense of Levinson):** All solutions are eventually trapped in a bounded absorbing set

- **Asymptotic compactness:** Global in time solutions are precompact with respect to the time shifts; they approach a compact $\omega-$limit set as $t \to \infty$

Long–time behavior, closed systems

Total energy

$$E(\varrho, \vartheta, \mathbf{u}) = \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta)$$

Dichotomy for the closed systems

$$\mathbf{g} = \mathbf{g}(x)$$

Either

$$\mathbf{g} = \nabla_x G \ \Rightarrow \ \text{all solutions tend to a single equilibrium}$$

or

$$\mathbf{g} \neq \nabla_x G \ \Rightarrow \ \int_\Omega E(t, \cdot) \ \mathrm{d}x \to \infty \text{ as } t \to \infty$$

Dynamical systems

Dynamical system

$$\mathbf{U}(t,\cdot) : [0,\infty) \times X \to X$$

- **Closed system:** $\mathbf{U}(t,X_0) \to \mathbf{U}_\infty$ equilibrium solution as $t \to \infty$
- **Open system:** $$\frac{1}{T}\int_0^T F(\mathbf{U}(t,X_0))\mathrm{d}t \to \int_X F(X)\ \mathrm{d}\mu,\ \ T \to \infty$$

μ a.s. in X_0

Principal mathematical problems:

- **Low regularity of global in time solutions**
  Global in time solutions necessary. For many problems in fluid dynamics – Navier–Stokes or Euler system – only weak solutions available
- **Lack of uniqueness**
  Solutions do not, or at least are not known to, depend uniquely on the initial data. Spaces of trajectories: Sell, Nečas, Temam and others
- **Propagation of oscillations**
  Realistic systems are partly hyperbolic: propagation of oscillations "from the past", singularities

Weak formulation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \ \ \varrho|_{\Gamma_{\rm in}} = \varrho_B$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S} + \varrho \mathbf{g}, \ \ \mathbf{u}|_{\partial\Omega} = \mathbf{u}_B$$

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) \boxed{\geq} \frac{1}{\vartheta} \left(\mathbb{S} : \mathbb{D}_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

ballistic energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{u}_B|^2 + \varrho e(\varrho, \vartheta) - \widetilde{\vartheta} \varrho s(\varrho, \vartheta) \right) \ \mathrm{d}x \ldots$$

Abstract setting

George Roger Sell 1937–2015

Space of entire trajectories

$$\mathcal{T} = C_{\rm loc}(R; X), \ \ t \in (-\infty, \infty)$$

ω**–limit set**

$$\omega[\mathbf{U}(\cdot, X_0)] \subset \mathcal{T}$$

$$\omega[\mathbf{U}(\cdot, X_0)] = \left\{ \mathbf{V} \in \mathcal{T} \ \middle| \ \mathbf{U}(\cdot + t_n, X_0) \to \mathbf{V} \text{ in } \mathcal{T} \text{ as } t_n \to \infty \right\}$$

Necessary ingredients

- **Dissipativity** – ultimate boundedness of trajectories
- **Compactness** – in appropriate spaces

Strong and weak ergodic hypothesis

Krylov – Bogolyubov construction

$$T \mapsto \frac{1}{T} \int_0^T \delta_{\mathbf{U}(\cdot + t, X_0)} \mathrm{d}t \text{ – a family of probability measures on } \mathcal{T}$$

$$\text{tightness in } \mathcal{T} \ \Rightarrow \ T_n \mapsto \frac{1}{T_n} \int_0^{T_n} \delta_{\mathbf{U}(\cdot + t, X_0)} \mathrm{d}t \to \mu \in \mathcal{P}[\mathcal{T}]$$

$[\mathcal{T}, \mu]$ stationary statistical solution

Ergodic hypothesis $\Leftrightarrow$ μ is unique $\Rightarrow$ $T \mapsto \frac{1}{T} \int_0^T \delta_{\mathbf{U}(\cdot + t, X_0)} \mathrm{d}t \to \mu$

unique $\approx$ unique on $\omega[\mathbf{U}(\cdot, X_0)]$

Weak ergodic hypothesis

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \delta_{\mathbf{U}(\cdot + t, X_0)} \mathrm{d}t = \mu \text{ exists in the narrow sense in } \mathcal{P}[\mathcal{T}]$$

$[\mathcal{T}, \mu]$ stationary statistical solution

Global bounded trajectories

Global in time weak solutions
$\mathbf{U} = [\varrho, \mathbf{m} = \varrho \mathbf{u}, S = \varrho s]$ – weak solution of the Navier–Stokes–Fourier system satisfying ballistic energy balance and defined for $t > T_0$

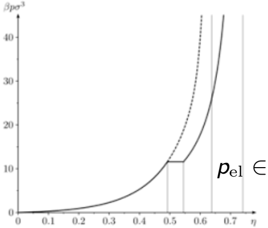
Bounded energy

$$\limsup_{t \to \infty} \int_{\Omega} E(\varrho, \mathbf{m}, S) \, \mathrm{d}x \leq \mathcal{E}_{\infty}$$

Available

- **Existence:** E.F. and A. Novotný, *Commun. Math. Phys.* 2021
  N. Chaudhuri and E.F. (Dirichlet b.c. for the temperature) *Preprint* 2021
- **Globally bounded solutions:** F. Fanelli, E. F., and M. Hofmanová **arxiv preprint No. 2006.02278**, 2020
  J. Březina, E. F., and A. Novotný, Communications in PDE's 2020
  E.F. , A. Novotný, M. Petcu – book in preparation

Hard sphere pressure EOS



$$p(\varrho, \vartheta) = p_{\rm el}(\varrho) + p_{\rm m}(\varrho, \vartheta) + p_{\rm rad}(\vartheta)$$

$$p_{\rm m} \approx \varrho\vartheta, \ P_{\rm rad} \approx \vartheta^4$$

$$p_{\rm el} \in C[0, \overline{\varrho}) \cap C^1(0, \overline{\varrho}), \ p'_{\rm el}(\varrho) > 0 \text{ for } \varrho > 0, \ \lim_{\varrho \to \overline{\varrho}-} p_{\rm el}(\varrho) = \infty$$

Ultimate boundedness of trajectories – bounded absorbing set

$$\limsup_{t \to \infty} \int_{\Omega} E(\varrho, \mathbf{m}, S) \ {\rm d}x \leq \mathcal{E}_\infty$$

$\mathcal{E}_\infty$ – universal constant

ω – limit sets

Trajectory space

$$X = \left\{ (\varrho, \mathbf{m}, S) \;\middle|\; \varrho(t,\cdot) \in L^{\gamma}(\Omega),\ \mathbf{m}(t,\cdot) \in L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d) \hookrightarrow W^{-k,2} \right.$$

$$\left. S(t,\cdot) \in \mathcal{M}(\Omega) \right\}$$

$$\mathcal{T} = C_{\rm loc}(R; L^1 \times W^{-k,2}) \times D_{\rm loc}(R; W^{-k,2})$$

Fundamental result on compactness [Fanelli, EF, Hofmanová, 2020]

The ω–limit set $\omega[\varrho, \mathbf{m}, S]$ of each global in time trajectory with globally bounded energy is:

- *non – empty*
- *compact* in $\mathcal{T}$
- time shift invariant
- consists of entire (defined for all $t \in R$) weak solutions of the Navier–Stokes–Fourier system

Propagation of oscillations

Equation of continuity

$$\partial_t \varrho + \mathbf{u} \cdot \nabla_x \varrho = -\varrho \mathrm{div}_x \mathbf{u}$$

Renormalized equation of continuity

$$\partial_t b(\varrho) + \mathrm{div}_x (b(\varrho)\mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} = 0$$

Weak convergence

$$b(\varrho_n) \to \overline{b(\varrho)} \text{ weakly in } L^1$$

$$\partial_t \Big[\overline{b(\varrho)} - b(\varrho) \Big] + \mathrm{div}_x \left(\overline{b(\varrho)\mathbf{u}} - b(\varrho)\mathbf{u} \right)$$

$$= \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} - \overline{\Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u}}$$

$$\Big[\overline{b(\varrho)} - b(\varrho) \Big](0, \cdot) = 0 \text{ is needed!}$$

Vanishing oscillation defect, I

Compactness of densities:

$$\varrho_n \equiv \varrho(\cdot + T_n) \to \varrho \text{ in } C_{\rm weak,loc}(R; L^\gamma(\Omega))$$
$$\varrho_n \log(\varrho_n) \to \overline{\varrho \log(\varrho)} \geq \varrho \log(\varrho)$$
$$\textbf{oscillation defect: } D(t) \equiv \int_\Omega \overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \ {\rm d}x \geq 0$$

Renormalized equation:

$$\frac{\rm d}{{\rm d}t} D + \int_\Omega \left[\overline{\varrho {\rm div}_x \mathbf{u}} - \varrho {\rm div}_x \mathbf{u} \right] \ {\rm d}x = 0, \ 0 \leq D \leq \overline{D}, \ t \in R$$

Lions' identity

$$\overline{\varrho {\rm div}_x \mathbf{u}} - \varrho {\rm div}_x \mathbf{u} = \overline{p(\varrho, \vartheta) \varrho} - \overline{p(\varrho, \vartheta)} \ \varrho \geq 0$$

Vanishing oscillation defect, II

Crucial differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}D + \Psi(D) \leq 0,\ 0 \leq D \leq \overline{D},\ t \in R$$

$$\Psi \in C(R),\ \Psi(0) = 0,\ \Psi(Z)Z > 0 \text{ for } Z \neq 0$$

$$\Rightarrow$$

$$D \equiv 0$$

Statistical stationary solutions

Application of Krylov – Bogolyubov method

$$\frac{1}{T_n}\int_0^{T_n} \delta_{\varrho(\cdot+t,\cdot),\mathbf{m}(\cdot+t,\cdot),S(\cdot+t,\cdot)}\ \mathrm{d}t \to \mu \in \mathcal{P}[\mathcal{T}] \text{ narrowly}$$

$[\mathcal{T},\mu]$ (canonical representation) – statististical stationary solution

$\mu(t)|_X$ (marginal) independent of $t \in R$

Application of Birkhoff – Khinchin ergodic theorem

$$\frac{1}{T}\int_0^{T} F(\varrho(t,\cdot),\mathbf{m}(t,\cdot),S(t\cdot))\mathrm{d}t \to \overline{F} \text{ as } T \to \infty$$

F bounded Borel measurable on X for $\mu -$ a.a. $(\varrho,\mathbf{m}) \in \omega$