The Witten Conjecture for homology

$S^1 \times S^3$

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June 23, 2021
1950s (Yang and Mills) Non-abelian version of the classical electromagnetic theory

1980s (Donaldson) Applications to topology of smooth 4-manifolds via the Donaldson polynomials

1988 (Witten) Quantum Field Theory: Donaldson polynomials = expectation values of certain observables. Degree-zero Donaldson polynomial = partition function.

1994 (Witten) The Seiberg–Witten invariants and the Witten Conjecture

**Natural domain:** closed oriented smooth 4-manifolds $X$ with $\pi_1(X) = 1$ and $b_2^+(X) > 1$ (Feehan and Leness)

**Extension:** smooth 4-manifolds $X$ with $b_2^+(X) = 1$ via wall-crossing formulas

**This project:** smooth 4-manifolds $X$ with $b_1(X) = 1$ and $b_2(X) = 0$
**Homology** $S^1 \times S^3$ is a smooth oriented closed spin 4-manifold $X$ such that

$$H_*(X; \mathbb{Z}) = H_*(S^1 \times S^3; \mathbb{Z}).$$

**Example.** A product $X = S^1 \times Y$, where $Y$ is an integral homology sphere.

**Example.** A “furled up” homology cobordism from $Y$ to itself:

![Diagram](image.png)

**Homology orientation** of $X$ is a choice of generator $1 \in H^1(X; \mathbb{Z})$. 
Donaldson Theory

Given a metric on $X$, consider connections $A$ in the trivial $SU(2)$ bundle on $X$ such that

$$F_A + *F_A = 0.$$ 

These are called instantons. The moduli space $\mathcal{M}(X, g)$ of irreducible instantons is compact, oriented, and 0-dimensional (perhaps after perturbation).

Furuta–Ohta invariant

$$\lambda_{FO}(X) = \frac{1}{4} \# \mathcal{M}(X, g).$$

**Theorem.** This is a well-defined diffeomorphism invariant of $X$ if $H_*(\tilde{X}; \mathbb{Q}) = H_*(S^3; \mathbb{Q})$, where $\tilde{X} \to X$ is the universal abelian cover.

The instantons in question are flat so $\lambda_{FO}(X)$ can be viewed as a count of irreducible representations $\pi_1(X) \to SU(2)$.

**Product case:** $\lambda_{FO}(S^1 \times Y) = \lambda(Y)$. 
Seiberg–Witten Theory

Given a metric $g$ on $X$ and a form $\beta \in \Omega^1(X, i\mathbb{R})$, consider the triples

$$(A, s, \varphi) \in \Omega^1(X, i\mathbb{R}) \times \mathbb{R}_{\geq 0} \times C^\infty(S^+)$$

such that

$$\begin{align*}
F_A^+ - s^2 \cdot \tau(\varphi, \varphi) &= d^+ \beta \\
D_A^+(X, g)(\varphi) &= 0, \quad \|\varphi\|_{L^2(X)} = 1
\end{align*}$$

Seiberg–Witten moduli space $M(X, g, \beta)$: the gauge equivalence classes of solutions of the above system. The solutions with $s = 0$ are called reducible.

**Theorem.** For generic $(g, \beta)$, the moduli space $M(X, g, \beta)$ is a compact oriented 0-dimensional manifold with no reducibles.

Denote by $\# M(X, g, \beta)$ the signed count of points in this moduli space. It depends on $g$ and $\beta$. 
Correction Term

Let $Y \subset X$ be a Poincaré dual to $1 \in H^1(X; \mathbb{Z})$ and cut $X$ open along $Y$ to obtain a cobordism $W$ from $Y$ to itself.

**End-periodic manifold** is a smooth manifold $Z_+ = Z \cup W \cup W \cup \ldots$ where $Z$ is a compact smooth spin 4-manifold with $\partial Z = Y$

![Diagram of a cobordism](image)

**Product case:** $X = S^1 \times Y$ gives rise to $Z_+ = Z \cup ([0, +\infty) \times Y)$. The index theory was studied by Atiyah, Patodi and Singer.

**General case:** the basics of index theory on $Z_+$ were established by Taubes. We developed this theory far enough to prove the following two theorems.
**Theorem** (Mrowka, Ruberman, S, 2011) For generic \((g, \beta)\), the operator

\[ D^+(Z_+, g) + \beta : L^2_1(Z_+) \to L^2(Z_+) \]

is Fredholm, and

\[ w(X, g, \beta) = \text{ind}(D^+(Z_+, g) + \beta) + \text{sign}(Z)/8 \]

is independent of the choice of \(Z\) and the way \(g\) and \(\beta\) are extended over \(Z \subset Z_+\).

**Theorem** (Mrowka, Ruberman, S, 2011)

\[ \lambda_{SW}(X) = \#M(X, g, \beta) - w(X, g, \beta) \]

is a diffeomorphism invariant of \(X\).

**Product case:** Weimin Chen (1997) and Yuhan Lim (1999).
**Witten Conjecture**

**Conjecture** (Mrowka, Ruberman, S, 2011) Let $X$ be a homology $S^1 \times S^3$ such that $H_*(\tilde{X}; \mathbb{Q}) = H_*(S^3; \mathbb{Q})$. Then

$$\lambda_{FO}(X) = -\lambda_{SW}(X).$$

**Theorem** (Yuhan Lim, 1999) Let $Y$ be an integral homology sphere. Then

$$\lambda_{FO}(S^1 \times Y) = -\lambda_{SW}(S^1 \times Y).$$

**Theorem** (Jianfeng Lin, Ruberman, S, 2020) Let $Y$ be an integral homology sphere and $X$ the mapping torus of a diffeomorphism $\tau : Y \to Y$ generating a semi-free action of $\mathbb{Z}/n$. Then

$$\lambda_{FO}(X) = -\lambda_{SW}(X).$$

The proof uses, on the Furuta–Ohta side, the equivariant Casson invariant of Collin–S (1999) and Ruberman-S (2004)
Explicit formulas

If $\tau$ has fixed points: $Y' = Y/\tau$ is an integral homology sphere and the quotient map $Y \to Y'$ is the $n$–fold branched cover with branch set a knot $K$. Then $\lambda_{FO}(X) = -\lambda_{SW}(X)$ equals

$$n \cdot \lambda(Y') + \frac{1}{8} \sum \text{sign}^{m/n}(K')$$

(also proved by Langte Ma (2020) using a surgery formula for $\lambda_{SW}(X)$).

If $\tau$ has no fixed points: $Y' = Y/\tau$ is a homology lens space which can be obtained by $(n/q)$–surgery along a knot $K$ in an integral homology sphere $\Sigma$. Then $\lambda_{FO}(X) = -\lambda_{SW}(X)$ equals

$$n \cdot \lambda(\Sigma) + \frac{1}{8} \sum \text{sign}^{m/n}(K) + \frac{q}{2} \Delta''_K(1).$$
Floer-theoretic interpretation

Assume that the Poincaré dual to $1 \in H^1(X; \mathbb{Z})$ can be chosen to be a rational homology sphere $Y \subset X$. Cut $X$ open along $Y$ to obtain $W$.

**Theorem** (Jianfeng Lin, Ruberman, S, 2018)

$$-\lambda_{SW}(X) = h_{SW}(Y) + \text{Lef}(W_*),$$

where $h_{SW}(Y)$ is the Frøyshov invariant and $\text{Lef}(W_*)$ is the Lefschetz number in the reduced monopole Floer homology $HM_{\text{red}}(Y)$ of Kronheimer and Mrowka.

**Theorem** (Anvari, 2019) If $Y \subset X$ is an integral homology sphere,

$$\lambda_{FO}(X) = h_D(Y) + \text{Lef}(W_*),$$

where $h_D(Y)$ is the instanton Frøyshov invariant and $\text{Lef}(W_*)$ is the Lefschetz number in the reduced instanton Floer homology $I_{\text{red}}(Y)$ of Frøyshov.
Topological Applications

Stem from calculating \( \text{Lef}(W_*') \) for mapping cylinders of self-diffeomorphisms \( \tau : Y \to Y \).

Example. Akbulut cork \( W_0 \)

Smooth compact contractible 4-manifold with boundary \( Y \). The boundary admits involution \( \tau : Y \to Y \) exchanging the two link components.
**Theorem** (Akbulut, 1991) The involution $\tau$ does not extend to a diffeomorphism of $W_0$.

**Theorem** (Jianfeng Lin, Ruberman, S, 2018) The involution $\tau : Y \rightarrow Y$ does not extend to a diffeomorphism of any compact smooth 4-manifold $W$ with boundary $Y$ such that

$$H_*(W; \mathbb{Z}/2) = H_*(D^4; \mathbb{Z}/2).$$

Extended by Dai, Hedden, and Mallick (2020) to other involutions using the Floer-theoretic framework developed by Hendricks, Manolescu, and Zemke.
**Theorem** (Jianfeng Lin, Ruberman, S, 2018) Let $K \subset S^3$ be a Khovanov-thin knot and $Y$ its double branched cover. Then

$$h(Y) = \frac{1}{8} \text{sign}(K).$$

This is known as the Manolescu–Owens Conjecture. It was proved by Manolescu–Owens for all alternating knots and by Lisca for all quasi-alternating knots.
Why $\lambda_{SW}(X)$ is metric independent

For simplicity, consider the product case $X = S^1 \times Y$ only. Then

$$\lambda_{SW}(X) = \# \mathcal{M}(Y) + \frac{1}{2} \eta(Y) + \frac{1}{8} \eta_{\text{Sign}}(Y).$$

For any $\mu \in \mathbb{R}$ consider the moduli space $\mathcal{M}(Y, \mu)$ of triples $(A, \varphi)$ such that

$$F_A = \tau(\varphi, \varphi), \quad D_A(\varphi) = \mu \varphi.$$ 

A schematic depiction of $\mathcal{M}(Y) = \bigcup_{\mu} \mathcal{M}(Y, \mu)$
For any $t > 0$ a direct calculation shows that

$$
\# M(Y, 0) + \frac{1}{2} \eta(Y) = \lim_{t \to 0} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \# M^*(Y, \mu) \cdot t^{1/2} \cdot e^{-t\mu^2} \, d\mu.
$$

In particular,

$$
\# M(Y) + \frac{1}{2} \eta(Y)
$$

is a continuous function of the metric. But $\eta_{\text{Sign}}(Y)$ is also a continuous function of the metric. Since the sum

$$
\# M(Y) + \frac{1}{2} \eta(Y) + \frac{1}{8} \eta_{\text{Sign}}(Y)
$$

is an integer, it must be constant as a function of the metric.