On the linearity of order-isomorphisms

Bas Lemmens
University of Kent (UK)
A cone $C$ in a vector space $V$ induces a partial ordering $\leq$, where
\[ x \leq y \text{ if } y - x \in C. \]
Here $C$ is convex, $\lambda C \subseteq C$ for all $\lambda \geq 0$, and $C \cap (-C) = \{0\}$.

$X \subseteq V$ and $Y \subseteq W$, where $V$ and $W$ are partially ordered vector spaces.

$f : X \to Y$ is an order-isomorphism, if $f$ is a bijection and
\[ x \leq y \text{ if and only if } f(x) \leq f(y). \]
So $f$ and $f^{-1}$ are both order-preserving.

**Question** When are such maps affine?
History

The question goes back to the 1950’s and was motivated by special relativity.

Alexandrov and Ovčinnikova (1953): order-isomorphisms $f : C \to C$, where $C$ is the finite dimensional Lorentz cone are linear.

Alexandrov (1967) Extension to more general finite dimensional cones.


Noll and Schäffer extended Alexandrov’s work to infinite dimensions in a sequence of papers (1977-1979).
Not always linear!

The map

\[ f : x \mapsto x^3 \]

is a **nonlinear** order-isomorphism on \( \mathbb{R} \).

If \((V, C)\) is a partially ordered vector space, then \( W = V \oplus \mathbb{R} \) with cone \( C \times \mathbb{R}_{\geq 0} \) admits a nonlinear order-isomorphism.

Can replace \( \mathbb{R}_{\geq 0} \) by the cone of nonnegative functions in \( C(K) \).

**Question** Which partially ordered vector spaces \((V, C)\) admit a **nonlinear** order-isomorphism \( f : C \to C \)?

Recent progress on this question by Walsh for order-unit spaces
A subset $X$ of $V$ is called an **upper set** if for each $x \in X$ and $v \in C$ we have that $x + v \in X$.

Examples: $V$, $C$, or $\text{int} \ C$.

We **only** consider order-isomorphisms $f : X \to Y$, where $X \subseteq V$ and $Y \subseteq W$ are **upper sets**.

Things are **different** (more complicated) when $X$ or $Y$ is not an upper set!

$X$ and $Y$ order-intervals: Drnovšek, Molnar, Mori, Šemrl, Roelands and Wortel
Some useful concepts

A cone $C$ in $V$ is **Archimedean** if for each $x \in V$ and $y \in C$ we have that

$$nx \leq y \text{ for all } n \geq 1 \implies x \leq 0.$$  

An element $u \in C$ is called an **order-unit** if for each $x \in V$ there exists $\lambda \geq 0$ such that $x \leq \lambda u$.

$U \subseteq V$ is said to be **directed** if for each $x, y \in U$ there exists $z \in U$ with $x \leq z$ and $y \leq z$.

$(V, C)$ is **directed** if $V$ is a directed set, which is equivalent to $V = C - C$.

**Notation**

$$[x, z] = \{y \in V : x \leq y \leq z\} \quad (\text{order-interval})$$

$$[x, \infty) = x + C = \{x + v : v \in C\} \quad (\text{cone with apex } x).$$
Extreme rays

Given \( x \in C \) with \( x \neq 0 \) we let

\[
R_x = \{ \lambda x : \lambda \geq 0 \} \quad \text{(ray through } x\text{)}
\]

\( R_x \) is said to be an extreme ray if \( 0 \leq y \leq x \) implies \( y = \lambda x \) for some \( 0 \leq \lambda \leq 1 \).

We call \( x + R \) an extreme half-line if \( R \) is an extreme ray of \( C \).
Order theoretic characterisation

**Intuition** An order-isomorphism should map extreme half-lines onto extreme half-lines.

**Fact** $H = x + R$ is an extreme half-line if and only if $H$ is maximal among all subsets $G$ of $[x, \infty)$ with $x \in G$ satisfying

1. $G$ is directed.
2. For each $y \in G$, $[x, y]$ is totally ordered.
3. $G$ contains at least 2 distinct points.

**Corollary** An order-isomorphism $f : X \to Y$ maps an extreme half-line $x + R$ onto an extreme half-line $f(x) + S$. 
Lemma Let $R$ and $S$ be distinct extreme rays of $C$ and $f : X \to Y$ be an order-isomorphism. For $x \in X$, $r \in R$ and $s \in S$ we have that

$$f(x + r + s) - f(x + s) = f(x + r) - f(x).$$
Engaged extreme rays

Let $\mathcal{R}$ be the collection of all extreme rays of $C$.

**Definition** An extreme ray $R$ of $C$ is said to be **engaged** if

$$R \subseteq \text{Span}(\mathcal{R} \setminus \{R\}) = \text{Span}\{s: s \in S \text{ and } S \in \mathcal{R} \setminus \{R\}\}.$$ 

Otherwise, we say it is **disengaged**.

Let $\mathcal{R}_E$ be the collection of all engaged extreme rays.

In the Lorentz cone $C$ with $\dim C \geq 3$, all rays in the boundary of $C$ are extreme rays and engaged.

For $\mathbb{R}^n_+$ the extreme rays are $R_i = \{\lambda e_i: \lambda \geq 0\}$, which are all disengaged.
A consequence

Define

\[ [x, \infty)_{R_E} = \{ x+r_1+\cdots+r_k \in [x, \infty) : r_i \in R_i \cup (-R_i), \ R_i \ \text{engaged} \}. \]

**Theorem** If \( f : [x, \infty) \to [y, \infty) \) is an order-isomorphism, then \( f \) is affine on \([x, \infty)_{R_E}\).
Theorem Suppose $X \subseteq V$ and $Y \subseteq W$ are upper sets in Archimedean partially ordered vector spaces, where $(V, C)$ is directed and $f : X \to Y$ is an order-isomorphism. If

$$C = \{r_1 + \cdots + r_k : r_i \in R_i \text{ extreme ray for all } i\}$$

and each $R \in \mathcal{R}$ is engaged, then $f$ is affine.

Condition, $C = \{r_1 + \cdots + r_k : r_i \in R_i \text{ extreme ray for all } i\}$ holds if $\dim V < \infty$, but rarely in infinite dimensions.
Cones with too few extreme rays

The cone

\[ C([0, 1])_+ = \{ f \in C([0, 1]) : f \geq 0 \} \]

has no extreme rays.

The cone \( B(H)^{sa}_+ \) of positive semidefinite operators has extreme rays \( R_P = \{ \lambda P : \lambda \geq 0 \} \), where \( P \) is a rank-one projection.

If \( \dim H = \infty \), then \( B(H)^{sa}_+ \neq \text{Span}_+ \{ R_P : P \text{ rank-one projection} \} \)
inf’s and sup’s

**Idea** Order-isomorphisms preserve the inf’s and sup’s if they exist.

For $U \subseteq [a, \infty)$ and $x \in [a, \infty)$ we say that

\[ x = \inf U \text{ in } [a, \infty), \]

if $x \leq u$ for all $u \in U$, and if $z \in [a, \infty)$ is such that $z \leq u$ for all $u \in U$, then $z \leq x$.

Similarly, $x = \sup U \text{ in } [a, \infty)$ is $x$ is the least upper bound of $U$ in $[a, \infty)$.
Given \( U \subseteq [a, \infty) \) the **inf-sup-hull** of \( U \) in \([a, \infty)\) is the set

\[
\{x \in [a, \infty): x = \inf_{\alpha \in A} \sup_{\beta \in B} u_{\alpha \beta} \text{ in } [a, \infty), \text{ where } u_{\alpha \beta} \in U\}.
\]

**Theorem** (L., van Gaans, van Imhoff) Suppose \( X \subseteq V \) and \( Y \subseteq W \) are upper sets in Archimedean partially ordered vector spaces, where \((V, C)\) is directed and \( f: X \to Y \) is an order-isomorphism. If \( C \) equals the inf-sup-hull of \( [0, \infty)_{\mathcal{R}_E} \), then \( f \) is affine.
Bounded self-adjoint operators

**Theorem** (Molnar) If $H$ is a Hilbert space with $\dim H \geq 2$ and $X, Y \subseteq B(H)_{sa}$ are upper sets, then every order-isomorphism $f : X \to Y$ is affine.

Easy to show that $B(H)^+_{sa}$ is the inf-sup-hull (sup-hull) of the engaged extreme rays.

**Corollary** (Molnar) There is no order-isomorphism from $B(H)_{sa}$ onto $\text{int } B(H)^+_{sa}$.

Extension to atomic JBW-algebras without a type $I_1$ part, Roelands and van Imhoff (2020).
Positive homogeneity

**Question** Is there a natural condition so that all order-isomorphisms are linear?

$f : \text{int } C \rightarrow \text{int } K$ is **positively homogeneous** if $f(\lambda x) = \lambda f(x)$ for all $x \in \text{int } C$ and $\lambda > 0$.

**Theorem** (Schäffer) If $(V, C)$ and $(W, K)$ are order-unit spaces, then every positively homogeneous order-isomorphism $f : \text{int } C \rightarrow \text{int } K$ is linear.
Theorem (L., van Gaans, van Imhoff) Suppose \((V, C)\) and \((W, K)\) are Archimedean partially ordered vector spaces, where \((V, C)\) is directed and \(C\) is equal to the inf-sup-hull of

\[
\{ r_1 + \cdots + r_k \in C : r_i \in R_i \cup (-R_i) \text{ where } R_i \text{ is an extreme ray} \}.
\]

Then every positive homogeneous order-isomorphism \(f : C \rightarrow K\) is linear.

Results applies to \(\ell_p(\mathbb{N})\) spaces (these spaces have no order unit).
Another related result

**Theorem** (L., van Gaans, van Imhoff) Let \((V, C, u)\) and \((W, K, e)\) be order unit spaces and \(X \subseteq V\) and \(Y \subseteq W\) be upper sets. Suppose that the inf-sup-hull of \([0, \infty)_{R_E}\) has a non-empty intersection with \(\text{int} C\) and that either \((V, \| \cdot \|_u)\) or \((W, \| \cdot \|_e)\) is separable and complete. Then every order-isomorphism \(f : X \to Y\) is affine.
An example

Let \( V = C([0, 1] \cup [2, 3]) \oplus \mathbb{R} \) with cone

\[
C = \{(f, \lambda) : \|f\|_\infty \leq \lambda\}.
\]

Then \((V, C, (0, 1))\) is a complete separable order-unit space.

The four points

\[(\pm 1_{[0,1]}, 1) \quad \text{and} \quad (\pm 1_{[2,3]}, 1)\]

correspond to the extreme rays, which are all engaged, since

\[
(1_{[0,1]}, 1) + (-1_{[0,1]}, 1) = 2(0, 1) = (1_{[2,3]}, 1) + (-1_{[2,3]}, 1).
\]

As \((0, 1) \in \text{int } C\), the theorem applies.
Thank you for your attention