Rigidity of Roe algebras

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- A bounded linear operator $T \in \mathfrak{B}(\ell^2(X,\ell^2(\mathbb{N})))$ can be written in the matrix form $\{T_{x,y}\}_{x,y\in X}$ where $T_{x,y} \in \mathfrak{B}(\ell^2(\mathbb{N}))$ defined by $T_{x,y}\xi = (T(\delta_y \otimes \xi))(x)$ for $\xi \in \ell^2(\mathbb{N})$.

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Definition

The **Roe algebra** $C^*(X)$ is the norm closure of all finite propagation and locally compact operators in $\mathfrak{B}(\ell^2(X; \ell^2(\mathbb{N})))$. E.g. $C^*(\Gamma) \cong \ell^\infty(\Gamma, \mathfrak{K}(\ell^2(\mathbb{N}))) \rtimes_r \Gamma$.

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- Application in Index Theory such as coarse Baum-Connes conjecture and Novikov conjecture.

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- Roe algebras are coarsely invariant: they contain coarse geometric information of the underlying spaces.

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- A subspace Y in a metric space (X, d) is **sparse** if $Y = \bigsqcup_n Y_n$ where each Y_n is finite and $d(Y_n, Y_m) \to \infty$ as $n+m \to \infty$ and $n \neq m$.

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- $T \in \mathfrak{B}(\ell^2(X; \ell^2(\mathbb{N})))$ is a **ghost** if $\|T_{x,y}\|_{\mathfrak{B}(\ell^2(\mathbb{N}))} \to 0$ as $x, y \to \infty$.

Analytic criterion

- If all sparse subspaces of X contain no **block-rank-one** ghost projections in their Roe algebras, then the rigidity holds. [L-Špakula-Zhang, 2020]

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Definition

Let $(X, d) = \bigsqcup_n (X_n, d_n)$ be a sparse space. A projection $P \in \mathfrak{B}(\ell^2(X; \ell^2(\mathbb{N})))$ is called a **block-rank-one** projection if

$$P = \bigoplus_n P_n,$$

where $P_n = (\cdot, \xi_n)\xi_n$ is a rank-one projection in $\mathfrak{B}(\ell^2(X_n; \ell^2(\mathbb{N})))$.

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► **The associated probability measure** m_n on X_n given by $m_n(\{x\}) := \|\xi_n(x)\|^2_{\ell^2(\mathbb{N})}$ for each $x \in X_n$. Hence we obtain a sequence of finite probability metric spaces $\{(X_n, d_n, m_n)\}_n$.

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- P is a ghost iff $\{(X_n, d_n, m_n)\}_n$ is **ghostly** (i.e. $\lim_n \sup_{x \in X_n} m_n(x) = 0$).

Theorem (L-Špakula-Zhang, 2020)

*Let $P \in \mathfrak{B}(\ell^2(X;\ell^2(\mathbb{N})))$ be a block-rank-one projection and m_n the associated measure on X_n. Then $P \in C^*(X)$ iff $\{(X_n, d_n, m_n)\}_n$ is a sequence of **measured asymptotic expanders**.*

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Definition (L-Vigolo-Zhang, 2019)

A sequence of finite probability metric spaces $\{(X_n, d_n, m_n)\}_n$ is called **measured asymptotic expanders** if $\forall\ \alpha \in (0, \frac{1}{2}]$, $\exists\ c_\alpha > 0$ and $R_\alpha > 0$ such that $\forall\ n$ and $\forall\ A \subset X_n$ with $\alpha \leq m_n(A) \leq \frac{1}{2}$, then $m_n(\partial_{R_\alpha} A) > c_\alpha m_n(A)$ (where $\partial_{R_\alpha} A = \{x \in X_n \setminus A : d_n(x, A) \leq R_\alpha\}$).

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- When $c_\alpha \equiv c > 0$, we call it **Measured expanders**.

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- When $c_\alpha \equiv c > 0$, we call it **Measured expanders**.

- When $c_\alpha \equiv c > 0$ and m_n = counting measure on finite graphs V_n, we recover **Expander graphs**: $\exists\ c > 0\ \forall\ n$ and $\forall\ A \subset X_n$ with $0 < |A| \leq \frac{1}{2}|V_n|$, then $|\partial A| > c|A|$.

Outline of the proof

- Structure theorem: Measured asymptotic expanders can be ”nicely” approximated by measured expander graphs (V_n, E_n, m_n) with **bounded measure ratios** (i.e. If $u \sim_{E_n} v$ in V_n, then $s \cdot m_n(v) \leq m_n(u) \leq \frac{m_n(v)}{s}$ for some $0 < s < 1$).

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- (V_n, E_n, m_n) may **not** come from any reversible random walk. However, we construct ν_n st (V_n, E_n, ν_n) has a reversible random walk, and ν_n and m_n control each other.

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- The associated Laplacian operator $\Delta_n \in C^*(X)$ to (V_n, E_n, ν_n) has spectral gap at 0 in the spectrum. So $Q_n = \chi_{\{0\}}(\Delta_n) \in C^*(X)$ and $Q_n \to P$ up to a compact perturbation. Hence, $P \in C^*(X)$.

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Corollary (L-Špakula-Zhang, 2020)

*There exist metric spaces that do **not** coarsely embed into any L^p-space for $1 \leq p < \infty$, but the rigidity still holds.*

Thank you for your attention!