Pointwise ergodic theorems for non-conventional bilinear polynomial averages

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joint work with Ben Krause and Terry Tao

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Ergodic averages as a tool to detect recurrent points

A measure-preserving system $(X, \mathcal{B}(X), \mu, T)$ is a σ-finite measure space $(X, \mathcal{B}(X), \mu)$ endowed with a measurable mapping $T\colon X \to X$, which preserves the measure μ, i.e. $\mu(T^{-1}[E]) = \mu(E)$ for all $E \in \mathcal{B}(X)$.

Question: Can one understand how points in measure-preserving systems $(X, \mathcal{B}(X), \mu, T)$ return close to themselves under iteration of the mapping T?

- (Birkhoff's and von Neumann's ergodic theorems (1931)) For every $1 \leq p < \infty$ and every $f \in L^p(X)$ the averages

$$A_N f(x) := \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x), \qquad \text{for} \qquad x \in X.$$

  converge μ-almost everywhere on X and in $L^p(X)$ norms.
- If we set $f(x) = \mathbb{1}_E(x)$, then

$$A_N \mathbb{1}_E(x) = \frac{1}{N} \#\{0 \leq n < N : T^n x \in E\}.$$

- Norm or pointwise convergence of $A_N f$ can be used to reprove the famous Poincaré recurrence theorem: if $\mu(X) = 1$, and $\mu(E) > 0$, then

$$\mu(E \cap T^{-n}[E]) > 0 \qquad \text{for some} \qquad n \in \mathbb{N}.$$

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Suppose that $E \subseteq \mathbb{N}$ has a positive upper Banach density, which means that $\limsup_{|I|\to\infty} \frac{|E\cap I|}{|I|} > 0$, where I ranges over intervals of $\mathbb{N}$.

- Then for any $k \geq 2$, there exist infinitely many progressions,

$$\{x, x+n, x+2n, \ldots, x+kn\} \subset E.$$

- The $k = 2$ case $\{x, x+n, x+2n\}$ is due to Roth in 1953.

The departure point for the modern theory of multiple ergodic averages is Furstenberg's ergodic-theoretic proof of Szemerédi's theorem.

- Let $(X, \mathcal{B}, \mu, T)$ be a probability measure-preserving system $\mu(X) = 1$ and $E \in \mathcal{B}(X)$ with $\mu(E) > 0$ then for every $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that

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Non-conventional ergodic averages

Let $(X, \mathcal{B}(X), \mu, T)$ be a σ-finite measure-preserving system with an invertible map $T : X \to X$. Let $P_1, \ldots, P_k \in \mathbb{Z}[\mathrm{n}]$, and $f_1, \ldots, f_k \in L^\infty(X)$. Define the non-conventional polynomial multiple ergodic average by

$$A_N^{P_1,\ldots,P_k}(f_1,\ldots,f_k)(x) := \frac{1}{N}\sum_{n=1}^{N} f_1(T^{P_1(n)}x)\ldots f_k(T^{P_k(n)}x) \quad \text{for} \quad x \in X.$$

Norm or pointwise convergence for these multiple averages allows us to detect polynomial patterns via multiple polynomial recurrence results.

- Given polynomials $P_1, \ldots, P_k \in \mathbb{Z}[\mathrm{n}]$ each with zero constant term. Let $(X, \mathcal{B}, \mu, T)$ be a probability measure-preserving system and $E \in \mathcal{B}(X)$ with $\mu(E) > 0$ then there exists $n \in \mathbb{N}$ such that

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- The latter result was proved by Bergelson and Leibman, and it was used to deduce that the subsets of integers with nonvanishing Banach density contain polynomial patterns of the form

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The current state of the art

Let $(X, \mathcal{B}(X), \mu, T)$ be a probability measure-preserving system $\mu(X) = 1$. Let $P_1, \ldots, P_k \in \mathbb{Z}[\mathrm{n}]$, and $f_1, \ldots, f_k \in L^\infty(X)$.

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Bergelson's conjecture

One of the central open problems in pointwise ergodic theory is a conjecture of V. Bergelson formulated in the late 1980's / early 1990's.

Theorem (Bergelson's conjecture)

Let $\mathbb{G}$ be a nilpotent group of measure preserving transformations of a probability space $(X, \mathcal{B}(X), \mu)$. Let $P_{j,i} \in \mathbb{Z}[\mathrm{n}]$ be polynomials and $T_1, \ldots, T_d \in \mathbb{G}$ and $f_1, \ldots, f_m \in L^\infty(X)$. Does the limit of the averages

$$\frac{1}{N} \sum_{n=1}^{N} \prod_{j=1}^{m} f_j(T_1^{P_{j,1}(n)} \cdots T_d^{P_{j,d}(n)} x) \tag{2}$$

exist μ-almost everywhere on X as $N \to \infty$?

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- Bergelson and Leibman showed that $L^2(X)$ norm convergence for (2) may fail if $\mathbb{G}$ is solvable.
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Multi-dimensional ergodic theorem

Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space with a family of invertible commuting and measure-preserving transformations $T_1, \ldots, T_d$. Let $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_d) : \mathbb{Z}^k \to \mathbb{Z}^d$ be a polynomial mapping with integer coefficients. Define

$$\mathcal{A}_N^{\mathcal{P}} f(x) := \frac{1}{|\mathbb{B}_N|} \sum_{m \in \mathbb{B}_N} f\big(T_1^{\mathcal{P}_1(m)} T_2^{\mathcal{P}_2(m)} \ldots T_d^{\mathcal{P}_d(m)} x\big),$$

where $\mathbb{B}_N := \{m \in \mathbb{Z}^k : |m| \le N\}$ is a discrete Euclidean ball.

Theorem (M., Stein, and Trojan and Zorin–Kranich)

For every $p \in (1, \infty)$ and every $f \in L^p(X)$ there exists $f^* \in L^p(X)$ such that

$$\lim_{N \to \infty} \mathcal{A}_N^{\mathcal{P}} f(x) = f^*(x)$$

for μ-almost every $x \in X$, and in $L^p(X)$ norm.

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Recent contribution to the nilpotent setting

In joint project with Alex Ionescu, Ákos Magyar and Tomek Szarek we proved the following nilpotent result.

Theorem (M., Ionescu, Magyar, and Szarek (2021))

Let $(X, \mathcal{B}(X), \mu)$ be a σ-finite space and let $T_1, \ldots, T_d : X \to X$ be a family of invertible and measure preserving transformations satisfying

$$[[T_i, T_j], T_k] = \mathrm{Id} \qquad \text{for all} \quad 1 \le i \le j \le k \le d.$$

Then for every polynomials $P_1, \ldots, P_d \in \mathbb{Z}[\mathrm{n}]$ and every $f \in L^p(X)$ with $1 < p < \infty$ the averages

$$\frac{1}{N} \sum_{n=1}^{N} f(T_1^{P_1(n)} \cdots T_d^{P_d(n)} x)$$

converge for μ-almost every $x \in X$ and in $L^p(X)$ norm as $N \to \infty$.

- One can think that $T_1, \ldots, T_d$ belong to a nilpotent group of step two of measure preserving mappings of a σ-finite space $(X, \mathcal{B}(X), \mu)$.
- We are also working on the extension of this result to nilpotent groups of step k for any $k \ge 3$.

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Progress on $A_N^{\mathrm{n},P(\mathrm{n})}(f,g)(x) = \frac{1}{N}\sum_{n=1}^{N} f(T^n x) g(T^{P(n)} x)$

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Thirty years after Bourgain's pointwise bilinear ergodic theorem joint with Ben Kruse and Terry Tao we established the following theorem.

Theorem (M., Krause, and Tao, (2020))

Let $(X, \mathcal{B}(X), \mu, T)$ be an invertibe σ-finite measure-preserving system, let $P \in \mathbb{Z}[\mathrm{n}]$ with $\deg(P) \geq 2$, and let $f \in L^{p_1}(X)$ and $g \in L^{p_2}(X)$ for some $p_1, p_2 \in (1, \infty)$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} \leq 1$.

(i) **(Mean ergodic theorem)** *The averages $A_N^{\mathrm{n},P(\mathrm{n})}(f,g)$ converge in $L^p(X)$ norm.*

(ii) **(Pointwise ergodic theorem)** *The averages $A_N^{\mathrm{n},P(\mathrm{n})}(f,g)$ converge pointwise almost everywhere.*

(iii) **(Maximal ergodic theorem)** *One has*

$$\|\sup_{N\in\mathbb{N}} |A_N^{\mathrm{n},P(\mathrm{n})}(f,g)|\|_{L^p(X)} \lesssim_{p_1,p_2,P} \|f\|_{L^{p_1}(X)} \|g\|_{L^{p_2}(X)}.$$

Key ideas

The proof is quite intricate, and relies on several deep results in the literature, including:

- the Ionescu–Wainger multiplier theorem (discrete Littlewood–Paley theory and paraproduct theory)
- the inverse theory of Peluse and Prendeville;
- Hahn–Banach separation theorem;
- L^p-improving estimates of Han–Kovač–Lacey–Madrid–Yang (derived from the Vinogradov mean value theorem).
- Rademacher–Menshov argument combined with Khinchine's inequality;
- $L^p(\mathbb{R})$ bounds for a shifted square function;
- bounded metric entropy argument from Banach space theory;
- van der Corput type estimates in the p-adic fields.

Inverse theorem of Peluse and Prendiville

An important ingredient in the proof is the inverse theorem of Peluse, which can be thought of as a counterpart of Weyl's inequality:

Theorem (Peluse, (2019/2020))

Let $m \geq 2$ *and* $P_1, \ldots, P_m \in \mathbb{Z}[\mathrm{n}]$ *each having zero constant term such that* $\deg P_1 < \ldots < \deg P_m$*. Let* $N \in \mathbb{N}$ *and* $\delta \in (0,1)$ *and assume that functions* $f_0, f_1, \ldots, f_m : \mathbb{Z} \to \mathbb{C}$ *are supported on* $[-N_0, N_0]$ *for some* $N_0 \simeq N^{\deg P_m}$*, and* $\|f_0\|_{L^\infty(\mathbb{Z})}, \|f_1\|_{L^\infty(\mathbb{Z})}, \ldots, \|f_m\|_{L^\infty(\mathbb{Z})} \leq 1$*, and suppose that*

$$\Big\| \frac{1}{N} \sum_{n=1}^{N} f_0(x) f_1(x - P_1(n)) \cdots f_m(x - P_m(n)) \Big\|_{L^1_x(\mathbb{Z})} \geq \delta N^{\deg P_m}.$$

Then there are $q, N' \in \mathbb{N}$ *satisfying* $1 \leq q \lesssim \delta^{-O(1)}$ *and* $\delta^{O(1)} N^{\deg P_1} \lesssim N' \leq N^{\deg P_1}$ *such that*

$$\Big\| \frac{1}{N'} \sum_{y=1}^{N'} f_1(x + qy) \Big\|_{L^1_x(\mathbb{Z})} \gtrsim \delta^{O(1)} N^{\deg P_m}$$

provided that $N \gtrsim \delta^{-O(1)}$.

Quantitative polynomial Szemerédi's

- Let $r_{P_1,\ldots,P_m}(N)$ denote the size of the largest subset of $\{1,\ldots,N\}$ containing no configuration of the form $x, x+P_1(n),\ldots,x+P_m(n)$ with $n \neq 0$. Berglson and Leibman showed proving polynomial multiple recurrence theorem that

$$r_{P_1,\ldots,P_m}(N) = o_{P_1,\ldots,P_m}(N),$$

  whenever $P_1,\ldots,P_m \in \mathbb{Z}[\mathrm{n}]$ and each having zero constant term.

- While quantitative bounds in Szemerédi's theorem for all $m \in \mathbb{N}$ are known due to work of Gowers, no bounds were known in general for the polynomial Szemerédi's theorem until a series of papers of Peluse and Prendiville.

- Peluse showed that there is a constant $\gamma_{P_1,\ldots,P_m} > 0$ such that

$$r_{P_1,\ldots,P_m}(N) \lesssim_{P_1,\ldots,P_m} \frac{N}{(\log\log N)^{\gamma_{P_1,\ldots,P_m}}}$$

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Bergelson's conjecture for commuting transformations

With Ben Krause and Sarah Peluse and Jim Wright we are also trying to understand the following problem:

Ultimate goal

Let $(X, \mathcal{B}(X), \mu)$ be a probability space equipped with commuting invertible measure-preserving maps $T_1, \ldots, T_k : X \to X$. Consider $P_1, \ldots, P_k \in \mathbb{Z}[\mathrm{n}]$ with distinct degrees and $f_1, \ldots, f_k \in L^\infty(X)$. It is expected that the averages

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Thank You!