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Transitions between configurations

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In this talk I would like to briefly review some constructions which can be used to obtain new configurations from old configurations.

Some of such constructions go back to as early as the 2nd half of the 19th century, but my aim is much more modest, and I restrict myself to developments in the last several decades.

Even in this case it is far from being a complete survey. All these constructions will be presented through illustrating examples, rather than, or besides, by a formal description.

Part of the results mentioned here were obtained in joint work with Tomo Pisanski, Leah Berman, Nino Baštić, Marko Boben, and Jurij Kovič.
Binary operations
Binary operations – (1) Incidence sum

- Introduced formally with this name by **GG & Pisanski, 2014**, but used intuitively by several other authors, too.
- Informally speaking, it is a superposition of two configurations, supplemented by new incidences.

**Definition**

By the incidence sum of configurations $C_1$ and $C_2$ we mean the configuration $C$ which is the disjoint union of $C_1$ and $C_2$, together with a specified set $I \subseteq P_1 \times L_2 \cup P_2 \times L_1$ of incident point-line pairs, where $P_i$ denotes the point set and $L_i$ denotes the line set of $C_i$, for $i = 1, 2$. We denote it by $C_1 \oplus C_2$. 

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Transitions between configurations

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Example 1. [Boben, GG & Pisanski, 2015]

Complete quadrilateral \((6_2, 4_3)\)
Example 1. [Boben, GG & Pisanski, 2015]

Complete quadrangle \((4_3, 6_2)\)
Example 1. [Boben, GG & Pisanski, 2015]

Desargues configuration: \((10_3) \cong (6_2, 4_3) \oplus (4_3, 6_2)\)
Example 1. [Boben, GG & Pisanski, 2015]

Desargues configuration: \((10_3) \simeq (6_2, 4_3) \oplus (4_3, 6_2)\)
Example 2. [Berman & Ng, 2010]

$C$: the 2-astral configuration $12\#(4, 1; 4, 5)$; type: $(24_4)$
Example 2. [Berman & Ng, 2010]

$C'$: a homothetic copy of $C$
Example 2. [Berman & Ng, 2010]

$C \oplus C'$. Type $(48_5)$
Example 3. [GG, 2018]

\[
\binom{3}{2}, \binom{3}{1} \quad \binom{4}{2}, \binom{4}{3} \quad \binom{4}{3}, \binom{4}{2}
\]

\[
\binom{5}{2}, \binom{5}{4} \quad \binom{5}{3}, \binom{5}{2} \quad \binom{5}{4}, \binom{5}{3}
\]

\[
\binom{6}{2}, \binom{6}{5} \quad \binom{6}{3}, \binom{6}{4} \quad \binom{6}{4}, \binom{6}{3} \quad \binom{6}{5}, \binom{6}{4}
\]

\[
\binom{7}{2}, \binom{7}{6} \quad \binom{7}{3}, \binom{7}{5} \quad \binom{7}{4}, \binom{7}{3} \quad \binom{7}{5}, \binom{7}{4} \quad \binom{7}{6}, \binom{7}{5}
\]

... 

Pascal’s triangle of Desargues–Cayley–Danzer configurations
Example 3. [GG, 2018]

\[
\begin{pmatrix}
\binom{3}{2}, \binom{3}{1} \\
\binom{4}{2}, \binom{4}{1} \\
\binom{5}{2}, \binom{5}{1} \\
\binom{6}{2}, \binom{6}{1} \\
\vdots
\end{pmatrix}
\]

“Cayley–Dickson line” [Saniga, Holweck & Pracna, 2015]
The notion of a **Cartesian product** of configurations has been introduced independently

- on the combinatorial level by *Pisanski & Servatius, 2013*, and
- for geometric point-line configurations by *GG, 2014*. 
Definition

Let $C$ be configuration of type $(v_r, b_k)$ and $C'$ a configuration of type $(v'_r, b'_k)$. Observe that these two configurations have the same number $k$ of points in each block. The **Cartesian product** of $C$ and $C'$ is a configuration of type

$$( (vv')(r+r'), (vb' + v'b)_k ),$$

whose point set is the Cartesian product of the point sets of $C$ and $C'$ and where there is a block incident to two points $(x, x')$ and $(y, y')$ if and only if either $x = y$ and there is a block incident to $x'$ and $y'$ in $C'$, or $x' = y'$ and there is a block incident to $x$ and $y$ in $C$. 
If the factors $C$, $C'$ are embedded in space of dimension $d$ and $d'$, then the product occurs in space of dimension $d + d'$. 

Example. An infinite sequence based on complete $n$-laterals:

- complete 5-lateral: \((10^2, 5^4) = (100^4) \subset P_4\)
- complete 7-lateral: \((21^2, 7^6) = (9261^6) \subset P_6\)
- complete 9-lateral: \((36^2, 9^8) = (167961^8) \subset P_8\)
- complete 11-lateral: \((55^2, 11^{10}) = (503284375^{10}) \subset P_{10}\)
If the factors $C, C'$ are embedded in space of dimension $d$ and $d'$, then the product occurs in space of dimension $d + d'$.

By repeated application, **Cartesian powers** can also be formed.
Binary operations – (2) Cartesian product

- If the factors \( C, C' \) are embedded in space of dimension \( d \) and \( d' \), then the product occurs in space of dimension \( d + d' \).
- By repeated application, Cartesian powers can also be formed.

**Example.** An infinite sequence based on complete \( n \)-laterals:

- complete 5-lateral: \((10_2, 5_4)^2 = (100_4) \subset \mathbb{P}^4\)
- complete 7-lateral: \((21_2, 7_6)^3 = (9261_6) \subset \mathbb{P}^6\)
- complete 9-lateral: \((36_2, 9_8)^4 = (1679616_8) \subset \mathbb{P}^8\)
- complete 11-lateral: \((55_2, 11_{10})^5 = (503284375_{10}) \subset \mathbb{P}^{10}\)

\[ \vdots \]

(A complete \( n \)-lateral is a configuration consisting of \( n \) lines in general position and of their \( \binom{n}{2} \) intersection points.)
Binary operations – (2) Cartesian product

For plane configurations, it can also be realized in the same plane, by means of the Minkowski addition [GG, Bašić, Kovič & Pisanski]:

- the point set of the product is the Minkowski sum of the point sets of the component configurations:

\[ P(C_1 \otimes C_2) = \{ x + y \mid x \in P(C_1), y \in P(C_2) \}. \]

- the new blocks are obtained as translates of the blocks of the one component by the position vectors of the points of the other component, and vice versa:

\[ B(C_1 \otimes C_2) = \{ \tau_y B(C_1) \mid y \in P(C_2) \} \cup \{ \tau_x B(C_2) \mid x \in P(C_1) \}. \]
For plane configurations, it can also be realized in the same plane, by means of the Minkowski addition [GG, Bašić, Kovič & Pisanski]:

\[(4_2, 2_4) \quad \Rightarrow \quad 2 \quad \Rightarrow \quad (16_4)\]
Incidence switch

“Switch” from old to new incidences [Grünbaum, 2009; Pisanski & Servatius, 2013]
Incidence switch

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Example 1. Applied to the Fano configuration:

(7₃)

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Incidence switch

Switch from old to new incidences [Grünbaum, 2009; Pisanski & Servatius, 2013]

**Example 1.** Applied to the Fano configuration:

\[ 2 \times (7_3) \]
Incidence switch

Switch from old to new incidences [Grünebaum, 2009; Pisanski & Servatius, 2013]

Example 1. Applied to the Fano configuration:
Incidence switch

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Example 1. Applied to the Fano configuration:
Switch from old to new incidences [Grünbaum, 2009; Pisanski & Servatius, 2013]

**Example 1.** Applied to the Fano configuration:

\[(14_3) \text{ (Realizable!)}\]
Remark.

- In this example, we started from a non-realizable configuration, and we obtained a realizable new configuration.
Incidence switch

Remark.

- In this example, we started from a non-realizable configuration, and we obtained a realizable new configuration.
- The converse case also occurs: if a configuration is connected to an incidence theorem, then the incidence switch results in a non-realizable configuration.
Example 2a. Applied to the Miquel configuration

[GG, Bašić, Kovič & Pisanski, 2021]
Example 2a. Applied to the Miquel configuration

\[ [G G, \text{Bašić, Kovič & Pisanski, 2021}] \]
Example 2a. Applied to the Miquel configuration

[GG, Bašić, Kovič & Pisanski, 2021]
Example 2a. Applied to the Miquel configuration

[GG, Bašić, Kovič & Pisanski, 2021]
Example 2b. Applied as a quaternary operation

[GG, Bašić, Kovič & Pisanski, 2021]

(32_3, 24_4)
The “Grünbaum incidence calculus” is the common name of a collection of constructions elaborated by Branko Grünbaum;

- presented in detail in [Grünbaum, 2009] (and in some earlier works of him);
- discussed also in [Pisanski & Servatius, 2013] (they also coined the name “Grünbaum calculus”);
- some of them are modified and generalized in [Berman, GG & Pisanski, 2021].
Grünbaum incidence calculus

- **Parallel switch**: 
  \[(p_q, n_k) \rightarrow ((kp)_q, (kn)_k)\]
  - uses parallel translations of the starting configuration;
  - the incidence numbers \(q, k\) are preserved.

- **Parallel replication**:
  \[(p_q, n_k) \rightarrow ((kp)_q + 1, (kn)_k + p)\]
  - uses parallel translations of the starting configuration;
  - if \(q < k\), then \((k - q)\)-fold application yields a balanced configuration.

- **Affine switch**: starting from an \((m,k)\) configuration with independent \(p \geq 0\) and \(q \geq 1\) parallel lines, for each integer \(r\) with \(1 \leq r \leq p + q\), it produces a configuration of type \((m + r, k)\).
  - uses affine transforms of the starting configuration.

- **Affine replication**:
  \[(n_k) \rightarrow ((n + 2)_k)\]
  - uses affine transforms of the starting configuration.

- **Deleted union construction**:
  \[(m_k) + (n_k) \rightarrow ((m + n - 1)_k)\]
  - uses projective geometric relationships.
Grünbaum incidence calculus

- **Parallel switch**: \((p_q, n_k) \rightarrow ((kp)_q, (kn)_k)\)
  - uses parallel translations of the starting configuration;
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  - uses parallel translations of the starting configuration;
  - if \(q < k\), then \((k - q)\)-fold application yields a balanced configuration.

Deleted union construction: \((m_k) + (n_k) \rightarrow (m + n - 1)_k\); uses projective geometric relationships.
Grünbaum incidence calculus

- **Parallel switch:** $(p_q, n_k) \rightarrow ((kp)_q, (kn)_k)$
  - uses parallel translations of the starting configuration;
  - the incidence numbers $q, k$ are preserved.

- **Parallel replication:** $(p_q, n_k) \rightarrow ((kp)_{q+1}, (kn+p)_k)$
  - uses parallel translations of the starting configuration;
  - if $q < k$, then $(k - q)$-fold application yields a balanced configuration.

- **Affine switch:** starting from an $(m_k)$ configuration with independent pencils of $p \geq 0$ and $q \geq 1$ parallel lines, for each integer $r$ with $1 \leq r \leq p + q$, it produces a configuration of type $((k - 1)m + r)$.
  - uses affine transforms of the starting configuration.
Parallel switch: \((p_q, n_k) \rightarrow ((kp)_q, (kn)_k)\)
- uses parallel translations of the starting configuration;
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Affine replication: \((n_k) \rightarrow (((k + 2)n)_{k+1})\);
- uses affine transforms of the starting configuration.
Grünbaum incidence calculus

- **Parallel switch**: \((p_q, n_k) \rightarrow ((kp)_q, (kn)_k)\)
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- **Affine replication**: \((n_k) \rightarrow (((k + 2)n)_{k+1})\);
  - uses affine transforms of the starting configuration.

- **Deleted union construction**: \((m_k) + (n_k) \rightarrow ((m + n - 1)_k)\);
  - uses projective geometric relationships.
Combining the \textit{parallel switch} and the \textit{deleted union} construction

\[(21_4)\]
Combining the **parallel switch** and the **deleted union** construction

\[ 4 \times (21_4) \]
Combining the parallel switch and the deleted union construction
Combining the parallel switch and the deleted union construction
Combining the **parallel switch** and the **deleted union** construction

\[(84_4) + (21_4)\]
Combining the parallel switch and the deleted union construction
Combining the parallel switch and the deleted union construction
Example 1. Producing a pair of \((21_7)\) configurations \([GG, 2019]\)
Example 1. Producing a pair of $(21_7)$ configurations [GG, 2019]

Start from the Grünbaum–Rigby configuration of type $(21_4)$
Example 1. Producing a pair of \((21_7)\) configurations \([GG, 2019]\)

Start from the Grünbaum–Rigby configuration of type \((21_4)\)

An observation by Luis Montejano: conics can be circumscribed around suitable 7-tuples of points of this configuration.
Transitions: replacing the lines of a configuration by conics

Example 1. Producing a pair of $\binom{21}{7}$ configurations [GG, 2019]
Example 1. Producing a pair of \((21_7)\) configurations [GG, 2019]

Remark. Recall that the lines and points of the \((21_4)\) Grünbaum–Rigby configuration correspond to the axes and centres of the 21 harmonic homologies within the automorphism group of the Klein quartic

\[
x^3y + y^3z + z^3x = 0.
\]

Question. Can the conics in the two \((21_7)\) configurations be related in some direct way to the Klein quartic?
Example 2. A family of type $(27_8)$

Start from a point-line configuration of type $(27_4)$
Grüntbaum notation: $9\#(4, 3; 2, 3; 1, 3)$ [Grüntbaum, 2009]
Example 2. A family of type \((27_8)\)
Example 2. A family of type \((27_8)\)

\[(27_8) - E_1 H_2 E_3\]
Transitions: replacing the lines of a configuration by conics

**Example 2.** A family of type \((27^8)\)

\[(27^8) - E_1 H_2 H_4\]
Example 2. A family of type $(27_8)$
Example 2. A family of type $(27_8)$

$(27_8) - H_1 H_3 H_4$
Example 2. A family of type $(27_8)$

$(27_8) \cdot H_1 H_2 H_4$
Transitions: replacing the lines of a configuration by conics

Example 2. A family of type \((27_8)\)

\[(27_8)-H_1 H_2 E_3\]
Example 2. A family of type \((27_8)\)
“Transmutation”: changing the shape of the blocks in a configuration while preserving the incidences.
(Some examples in [GG and Pisanski, 2014].)
“Transmutation”: changing the shape of the blocks in a configuration while preserving the incidences.

(Some examples in [GG and Pisanski, 2014].)

Example. The $(21_4)$ Grünbaum–Rigby configuration.
Example. The $\text{(21}_4\text{)}$ Grünbaum–Rigby configuration: isometric point-circle representation.
Transitions: non-isomorphic transmutation

**Example:** Transition from type $(24_4)$ to type $(24_4, 12_8)$
**Example**: Transition from type \((24_4)\) to type \((24_4, 12_8)\)

- It can be combined with **Cartesian squaring**:

\[
\begin{align*}
(24_4) & \quad \rightarrow \quad (24_4, 12_8) \\
\downarrow & \quad \downarrow \\
(576_8, 1152_4) & \quad \rightarrow \quad (576_8)
\end{align*}
\]
Transitions: radical axes and centres of point-circle configurations

**Example**: an unexpected connection between the Miquel configuration and the Steiner–Plücker configuration.

Construction:
- start from the $(8_3, 6_4)$ Miquel configuration of points and circles;
- take the radical axis for each pair of circles;
- take the radical centre for each triple of circles.

We obtain a point-line configuration of type $(20_3, 15_4)$ which is isomorphic to the Steiner–Plücker configuration.
Example: an unexpected connection between the Miquel configuration and the Steiner–Plücker configuration.

The \((8_3, 6_4)\) Miquel configuration
Example: an unexpected connection between the Miquel configuration and the Steiner–Plücker configuration.
Example: an unexpected connection between the Miquel configuration and the Steiner–Plücker configuration.

The $(20_3, 15_4)$ Steiner–Plücker configuration
A generalization:

- start from a point-circle configuration of type \( ((kn)_3, (3n)_k) \) \((k, n \geq 3)\) such that no four circles have the same radical centre;
- the construction yields a point-line configuration of type \( \left( \binom{3n}{3} : \binom{3n}{2}^{3n-2} \right) \).

[2] Berman, L. W., Gévay, G. and Pisanski, T., Connected geometric \((n_k)\) configurations exist for almost all \(n\) (to be published in *Art Discrete Appl. Math.)*


Thank you for your attention.