Regularity of a weak solution to a linear fluid-composite structure interaction problem

Marija Galić

Department of Mathematics, Faculty of Science, University of Zagreb

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Outline

This talk will be divided into three sections:

- Problem description
- Existence result
- Regularity results

The first two parts are a joint work with:

- Sunčica Čanić, *University of California, Berkeley*,
- Boris Muha, *Faculty of Science, University of Zagreb*,
- Josip Tambača, *Faculty of Science, University of Zagreb*.

Matko Ljulj and Yifan Wang did the numerical simulations of the FSI problem considered in this talk.

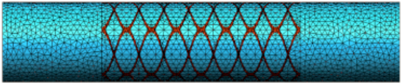
Introduction

- We consider a linear fluid-structure interaction problem between an incompressible, viscous, Newtonian fluid and the motion of an elastic structure.
- The fluid flow is modeled by the time-dependent Stokes equations while the structure is modeled as a linearly elastic cylindrical Koiter shell coupled with a net made of elastic rods.

fluid	3D Stokes equations
shell	2D linear Koiter shell
mesh	1D net made of elastic rods

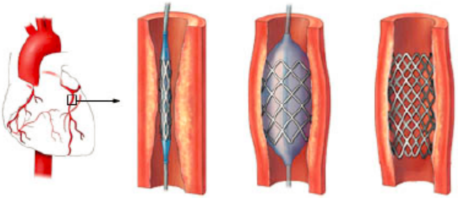
Main assumptions of the model

- The problem is set on a cylindrical domain in 3D, and is driven by the time-dependent inlet and outlet pressure data.
- The flow is assumed to be laminar, and the structure displacement is assumed to be small allowing displacement in all three spatial directions.
- No smallness on the structure velocity is assumed.
- The fluid and the mesh-supported structure are coupled via the kinematic and dynamic coupling conditions describing continuity of velocity and balance of contact forces.



Motivation

- This problem was motivated by a study of blood flow through medium-to-large human arteries, such as the aorta or coronary arteries, treated with vascular stents.
- The vascular stent is a thin, metallic mesh tube which is inserted at the location of the narrowing of a diseased coronary artery in order to prop the artery open.
- The procedure of inserting the stent inside the artery is called coronary angioplasty.



Model description - the fluid

We consider the flow of an incompressible, viscous fluid through a cylindrical domain, denoted by Ω:

$$\Omega = \{(z, x, y) \in \mathbb{R}^3 : z \in (0, L), \sqrt{x^2 + y^2} \leq R\}.$$

The fluid domain boundary consists of three parts: the lateral boundary Γ, which is a cylinder of radius R, the inlet boundary Γ_{in} and the outlet boundary Γ_{out}. The time-dependent Stokes equations are used to model the flow in Ω:

Fluid

$$\left.\begin{array}{rcl} \rho_F \partial_t \mathbf{u} & = & \nabla \cdot \boldsymbol{\sigma}, \\ \nabla \cdot \mathbf{u} & = & 0, \end{array}\right\} \text{ in } \Omega, \quad t \in (0, T). \tag{1}$$

Model description - the fluid

At the inlet and outlet we prescribe the pressure, with the tangential fluid velocity equal to zero:

$$\left.\begin{array}{c} p = P_{in/out}(t), \\ \mathbf{u} \times \mathbf{e}_z = 0, \end{array}\right\} \text{ on } \Gamma_{in/out},$$

where $P_{in/out}$ are given.

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where $P_{in/out}$ are given.
The fluid velocity will be assumed to belong to the following classical function space:

Fluid space

$$V_F = \{\mathbf{u} \in H^1(\Omega;\mathbb{R}^3) : \nabla \cdot \mathbf{u} = 0, \mathbf{u} \times \mathbf{e}_z = 0 \text{ on } \Gamma_{in/out}\}.$$

Model description - the shell

The lateral boundary of the fluid domain will be assumed elastic, and modeled as a clamped cylindrical Koiter shell of thickness h, length L, and reference radius of the middle surface R. This reference configuration, which we denote by Γ, can be defined via parameterization

$$\boldsymbol{\varphi} : \omega \to \mathbb{R}^3, \quad \boldsymbol{\varphi}(z, \theta) = (z, R\cos\theta, R\sin\theta),$$

where $\omega = (0, L) \times (0, 2\pi)$, and $R > 0$. Under loading, the Koiter shell is displaced from its reference configuration Γ by a displacement $\boldsymbol{\eta} = \boldsymbol{\eta}(t, z, \theta) = (\eta_z, \eta_r, \eta_\theta)$. Let V_K denote the following function space:

Shell space

$$\begin{aligned} V_K = \{\boldsymbol{\eta} = (\eta_z, \eta_r, \eta_\theta) \in H^1(\omega) \times H^2(\omega) \times H^1(\omega) : \\ \boldsymbol{\eta}(t, z, \theta) = \partial_z \eta_r(t, z, \theta) = 0, z \in \{0, L\}, \theta \in (0, 2\pi), \\ \boldsymbol{\eta}(t, z, 0) = \boldsymbol{\eta}(t, z, 2\pi), \partial_\theta \eta_r(t, z, 0) = \partial_\theta \eta_r(t, z, 2\pi), z \in (0, L)\}. \end{aligned}$$

Model description - the shell

The displacement $\boldsymbol{\eta}(t,z,\theta) = (\eta_z, \eta_r, \eta_\theta)$ of the deformed shell from the reference configuration Γ is a solution to the following elastodynamics problem, written in weak form:

Koiter shell

find $\boldsymbol{\eta} \in V_K$ such that

$$\rho_K h \int_\omega \partial_t^2 \boldsymbol{\eta} \cdot \boldsymbol{\psi} R + \langle \mathcal{L}\boldsymbol{\eta}, \boldsymbol{\psi} \rangle = \int_\omega \mathbf{f} \cdot \boldsymbol{\psi} R, \quad \forall \boldsymbol{\psi} \in V_K. \tag{2}$$

Here, ρ_K is the shell density and $\mathbf{f}$ is the force density acting on the shell. $\mathcal{L}$ is an operator that describes elastic properties (change of metric tensor and change of curvature tensor) of the shell. We emphasize that we have the coercivity of the operator $\mathcal{L}$, i.e. $\langle \mathcal{L}\boldsymbol{\eta}, \boldsymbol{\eta} \rangle \geq c\|\boldsymbol{\eta}\|^2$, $\forall \boldsymbol{\eta} \in V_K$.

Model description - the mesh

An elastic mesh is a three-dimensional elastic body defined as a union of three-dimensional slender components called struts. Since each strut is "thin", meaning that its two dimensions are small comparing to the third one, we approximate it with one-dimensional curved rod model. For the $i-$th curved rod, the middle line is parameterized via

$$\mathbf{P}_i : [0, l_i] \to \boldsymbol{\varphi}(\overline{\omega}), \quad i = 1, \dots, n_E,$$

and on each rod we have next family of equations:

Mesh

$$\begin{aligned} \rho_S A_i \partial_t^2 \mathbf{d}_i &= \partial_s \mathbf{p}_i + \mathbf{f}_i, \\ \rho_S M_i \partial_t^2 \mathbf{w}_i &= \partial_s \mathbf{q}_i + \mathbf{t}_i \times \mathbf{p}_i, \\ 0 &= \partial_s \mathbf{w}_i - Q_i H_i^{-1} Q_i^T \mathbf{q}_i, \\ 0 &= \partial_s \mathbf{d}_i + \mathbf{t}_i \times \mathbf{w}_i. \end{aligned} \tag{3}$$

Model description - the mesh

Here, $\mathbf{d}_i$ is the displacement of the middle line of the $i-$th rod, $\mathbf{w}_i$ is the infinitesimal rotation of the cross-section of the $i-$th rod, $\mathbf{q}_i$ is the contact moment, and $\mathbf{p}_i$ is the contact force.

Model description - the mesh

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At each vertex of the mesh we need to prescribe coupling conditions:

- kinematic ... continuity of displacements and infinitesimal rotations
- dynamic ... balance of contact forces and contact moments

Model description - the mesh

We first introduce a function space consisting of all the H^1-functions $(\mathbf{d}, \mathbf{w})$ defined on the entire net $\mathcal{N}$, such that they satisfy the kinematic coupling conditions at each vertex:

$$H^1(\mathcal{N};\mathbb{R}^6) = \{(\mathbf{d},\mathbf{w}) = ((\mathbf{d}_1,\mathbf{w}_1),\ldots,(\mathbf{d}_{n_E},\mathbf{w}_{n_E})) \in \prod_{i=1}^{n_E} H^1(0,l_i;\mathbb{R}^6) :$$
$$\mathbf{d}_i(\mathbf{P}_i^{-1}(V)) = \mathbf{d}_j(\mathbf{P}_j^{-1}(V)), \mathbf{w}_i(\mathbf{P}_i^{-1}(V)) = \mathbf{w}_j(\mathbf{P}_j^{-1}(V)),$$
$$\forall V \in \mathcal{V}, V = e_i \cap e_j,\ i,j = 1,\ldots,n_E\}.$$

The solution space is defined to contain the conditions of inextensibility and unshearability as follows:

Mesh space

$$V_S = \{(\mathbf{d},\mathbf{w}) \in H^1(\mathcal{N};\mathbb{R}^6) : \partial_s \mathbf{d}_i + \mathbf{t}_i \times \mathbf{w}_i = 0, i = 1,\ldots,n_E\}.$$

Coupling between the shell and the mesh

The elastic mesh is fixed to the shell

$$\bigcup_{i=1}^{n_E} \mathbf{P}_i([0, l_i]) \subset \Gamma = \boldsymbol{\varphi}(\overline{\omega}).$$

Since $\boldsymbol{\varphi}$ is injective on ω, functions $\boldsymbol{\pi}_i$, denoting the reparameterizations of the mesh struts:

$$\boldsymbol{\pi}_i = \boldsymbol{\varphi}^{-1} \circ \mathbf{P}_i : [0, l_i] \to \overline{\omega}, \quad i = 1, \dots, n_E$$

are well defined.

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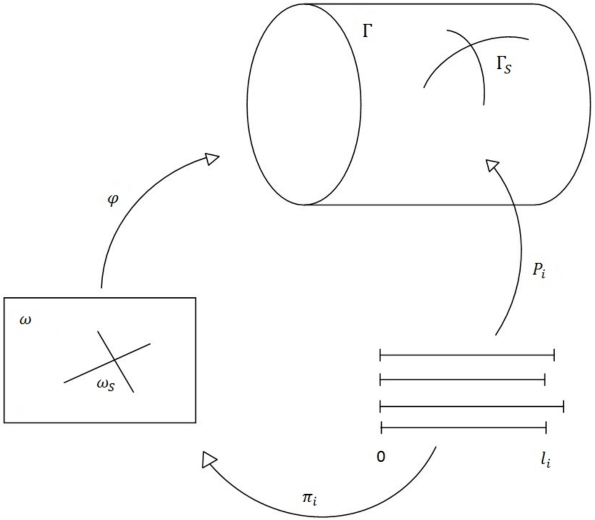
are well defined.

The elastic mesh and the shell are coupled through the following coupling conditions:

- kinematic: $\boldsymbol{\eta}(t, \boldsymbol{\pi}_i(s_i)) = \mathbf{d}_i(t, s_i), \forall s_i \in [0, l_i]$ such that $\boldsymbol{\pi}_i(s_i) = (z, \theta) \in \omega$,
- dynamic: $\mathbf{f}R = -\sum_{i=1}^{n_E} \frac{\mathbf{f}_i \circ \boldsymbol{\pi}_i^{-1}}{\|\boldsymbol{\pi}_i' \circ \boldsymbol{\pi}_i^{-1}\|} \delta_{J_i}, \forall (z, \theta) \in \omega$, where $J_i = \boldsymbol{\pi}_i([0, l_i])$.

Parameterization of the mesh struts



Coupling between the fluid and the structure

The coupling between the fluid and the structure is defined by two sets of coupling conditions: the kinematic and dynamic coupling conditions, satisfied at the fixed, lateral boundary Γ, giving rise to a linear fluid-structure coupling:

- kinematic: $\partial_t \boldsymbol{\eta} = \mathbf{u}|_\Gamma \circ \boldsymbol{\varphi}$ on $(0,T) \times \omega$,
- dynamic:

$$\rho_K h \partial_t^2 \boldsymbol{\eta} R + \mathcal{L}\boldsymbol{\eta} + \sum_{i=1}^{n_E} \frac{\mathbf{f}_i \circ \boldsymbol{\pi}_i^{-1}}{\|\boldsymbol{\pi}_i' \circ \boldsymbol{\pi}_i^{-1}\|} \delta_{J_i} = -J(\boldsymbol{\sigma} \circ \boldsymbol{\varphi})(\mathbf{n} \circ \boldsymbol{\varphi}) \text{ on } (0,T) \times \omega,$$

where J denotes the Jacobian of the transformation from cylindrical to Cartesian coordinates, and $\mathbf{n}$ denotes the outer unit normal on Γ.

The fluid-mesh-shell problem

In summary, we study the following fluid-structure interaction problem.
Problem 1. Find $(\mathbf{u}, p, \boldsymbol{\eta}, \mathbf{d}, \mathbf{w})$ such that

$$\left.\begin{array}{rcl} \rho_F \partial_t \mathbf{u} & = & \nabla \cdot \boldsymbol{\sigma} \\ \nabla \cdot \mathbf{u} & = & 0 \end{array}\right\} \text{ in } (0,T) \times \Omega, \quad (4)$$

$$\left.\begin{array}{rcl} \partial_t \boldsymbol{\eta} & = & \mathbf{u} \circ \boldsymbol{\varphi} \\ \rho_K h \partial_t^2 \boldsymbol{\eta} R + \mathcal{L}\boldsymbol{\eta} + \sum_{i=1}^{n_E} \frac{\mathbf{f}_i \circ \boldsymbol{\pi}_i^{-1}}{||\boldsymbol{\pi}_i' \circ \boldsymbol{\pi}_i^{-1}||} \delta_{J_i} & = & -J(\boldsymbol{\sigma} \circ \boldsymbol{\varphi})(\mathbf{n} \circ \boldsymbol{\varphi}) \end{array}\right\} \text{ on } (0,T) \times \omega, \quad (5)$$

$$\left.\begin{array}{rcl} \rho_S A_i \partial_t^2 \mathbf{d}_i & = & \partial_s \mathbf{p}_i + \mathbf{f}_i \\ \rho_S M_i \partial_t^2 \mathbf{w}_i & = & \partial_s \mathbf{q}_i + \mathbf{t}_i \times \mathbf{p}_i \\ 0 & = & \partial_s \mathbf{w}_i - Q_i H_i^{-1} Q_i^T \mathbf{q}_i \\ 0 & = & \partial_s \mathbf{d}_i + \mathbf{t}_i \times \mathbf{w}_i \end{array}\right\} \text{ on } (0,T) \times (0, l_i). \quad (6)$$

The fluid-mesh-shell problem

Problem (4)-(6) is supplemented with the following set of boundary and initial conditions:

$$\left\{\begin{array}{rcll} p & = & P_{in/out}(t), & \text{on } (0,T)\times\Gamma_{in/out}, \\ \mathbf{u}\times\mathbf{e}_z & = & 0, & \text{on } (0,T)\times\Gamma_{in/out}, \\ \boldsymbol{\eta}(t,0,\theta) & = & \boldsymbol{\eta}(t,L,\theta)=0, & \text{on } (0,T)\times(0,2\pi), \\ \partial_z\eta_r(t,0,\theta) & = & \partial_z\eta_r(t,L,\theta)=0, & \text{on } (0,T)\times(0,2\pi), \\ \boldsymbol{\eta}(t,z,0) & = & \boldsymbol{\eta}(t,z,2\pi), & \text{on } (0,T)\times(0,L), \\ \partial_\theta\eta_r(t,z,0) & = & \partial_\theta\eta_r(t,z,2\pi), & \text{on } (0,T)\times(0,L), \end{array}\right. \tag{7}$$

$$\begin{array}{c} \mathbf{u}(0)=\mathbf{u}_0,\ \boldsymbol{\eta}(0)=\boldsymbol{\eta}_0,\ \partial_t\boldsymbol{\eta}(0)=\mathbf{v}_0, \\ \mathbf{d}_i(0)=\mathbf{d}_{0i},\ \partial_t\mathbf{d}_i(0)=\mathbf{k}_{0i},\ \mathbf{w}_i(0)=\mathbf{w}_{0i},\ \partial_t\mathbf{w}_i(0)=\mathbf{z}_{0i}. \end{array} \tag{8}$$

Energy inequality

The formal energy estimate shows that the total energy $E(t)$ of the problem is bounded by the data of the problem

$$\frac{d}{dt}E(t) + D(t) \leq C(P_{in}(t), P_{out}(t)), \tag{9}$$

where $E(t)$ denotes the total energy of the coupled problem (the sum of the kinetic and elastic energy), $D(t)$ denotes dissipation due to fluid viscosity, and $C(P_{in}(t), P_{out}(t))$ is a constant which depends only on the L^2-norms of the inlet and outlet pressure data.

Definition of a weak solution

We define the following evolution spaces associated with the fluid problem, the Koiter shell problem, the mesh problem and the coupled mesh-shell problem:

- $V_F(0,T) = L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V_F)$,
- $V_K(0,T) = W^{1,\infty}(0,T;L^2(R;\omega)) \cap L^\infty(0,T;V_K)$,
- $V_S(0,T) = W^{1,\infty}(0,T;L^2(\mathcal{N})) \cap L^\infty(0,T;V_S)$,
- $V_{KS}(0,T) = \{(\boldsymbol{\eta}, \mathbf{d}, \mathbf{w}) \in V_K(0,T) \times V_S(0,T) : \boldsymbol{\eta} \circ \boldsymbol{\pi} = \mathbf{d}$ on $\prod_{i=1}^{n_E}(0,l_i)\}$.

Definition of a weak solution

The solution space for the coupled fluid-mesh-shell interaction problem involves the kinematic coupling condition, which is, thus, enforced in a strong way:

$$\mathcal{V}(0,T) = \{(\mathbf{u}, \boldsymbol{\eta}, \mathbf{d}, \mathbf{w}) \in V_F(0,T) \times V_{KS}(0,T) : \mathbf{u} \circ \boldsymbol{\varphi} = \partial_t \boldsymbol{\eta} \text{ on } \omega\}.$$

The associated test space is given by:

$$\mathcal{Q}(0,T) = \{(\boldsymbol{\upsilon}, \boldsymbol{\psi}, \boldsymbol{\xi}, \boldsymbol{\zeta}) \in C_c^1([0,T); V_F \times V_{KS}) : \boldsymbol{\upsilon} \circ \boldsymbol{\varphi} = \boldsymbol{\psi} \text{ on } \omega\}.$$

Definition of a weak solution

We say that $(\mathbf{u}, \boldsymbol{\eta}, \mathbf{d}, \mathbf{w}) \in \mathcal{V}(0,T)$ is a weak solution of Problem 1 if for all test functions $(\boldsymbol{\upsilon}, \psi, \boldsymbol{\xi}, \boldsymbol{\zeta}) \in \mathcal{Q}(0,T)$ the following equality holds:

$$
\begin{aligned}
&-\rho_F \int_0^T \int_\Omega \mathbf{u} \cdot \partial_t \boldsymbol{\upsilon} + 2\mu_F \int_0^T \int_\Omega \mathbf{D}(\mathbf{u}) : \mathbf{D}(\boldsymbol{\upsilon}) - \rho_K h \int_0^T \int_\omega \partial_t \boldsymbol{\eta} \cdot \partial_t \boldsymbol{\psi} R \\
&+ \int_0^T a_K(\boldsymbol{\eta}, \boldsymbol{\psi}) - \rho_S \sum_{i=1}^{n_E} A_i \int_0^T \int_0^{l_i} \partial_t \mathbf{d}_i \cdot \partial_t \boldsymbol{\xi}_i - \rho_S \sum_{i=1}^{n_E} \int_0^T \int_0^{l_i} M_i \partial_t \mathbf{w}_i \cdot \partial_t \boldsymbol{\zeta}_i \\
&+ \int_0^T a_S(\mathbf{w}, \boldsymbol{\zeta}) = \int_0^T \langle F(t), \boldsymbol{\upsilon} \rangle_{\Gamma_{in/out}} + \rho_F \int_\Omega \mathbf{u}_0 \cdot \boldsymbol{\upsilon}(0) + \rho_K h \int_\omega \mathbf{v}_0 \cdot \boldsymbol{\psi}(0) R \\
&+ \rho_S \sum_{i=1}^{n_E} A_i \int_0^{l_i} \mathbf{k}_{0i} \cdot \boldsymbol{\xi}_i(0) + \rho_S \sum_{i=1}^{n_E} \int_0^{l_i} M_i \mathbf{z}_{0i} \cdot \boldsymbol{\zeta}_i(0),
\end{aligned}
\tag{10}
$$

Existence of the weak solution

Theorem

Let $\mathbf{u}_0 \in L^2(\Omega)$, $\boldsymbol{\eta}_0 \in H^1(\omega)$, $\mathbf{v}_0 \in L^2(R;\omega)$, $(\mathbf{d}_0, \mathbf{w}_0) \in V_S$, $(\mathbf{k}_0, \mathbf{z}_0) \in L^2(\mathcal{N};\mathbb{R}^6)$ be such that

$$\nabla \cdot \mathbf{u}_0 = 0,\ (\mathbf{u}_0|_\Gamma \circ \boldsymbol{\varphi}) \cdot \mathbf{e}_r = (\mathbf{v}_0)_r,\ \mathbf{u}_0|_{\Gamma_{in/out}} \times \mathbf{e}_z = 0,\ \boldsymbol{\eta}_0 \circ \boldsymbol{\pi} = \mathbf{d}_0.$$

Furthermore, let all the physical constants be positive: $\rho_K, \rho_S, \rho_F, \lambda, \mu, \mu_F > 0$ and $A_i > 0, \forall i = 1, \dots, n_E$, and let $P_{in/out} \in L^2_{loc}(0,\infty)$. Then for every $T > 0$ there exists a weak solution to Problem 1.

Existence of the weak solution

In order to prove the existence of the weak solution to Problem 1, we proceed as follows:

- Use the Lie operator splitting scheme to split the problem into two subproblems, the fluid and the structure subproblem.

Existence of the weak solution

In order to prove the existence of the weak solution to Problem 1, we proceed as follows:

- Use the Lie operator splitting scheme to split the problem into two subproblems, the fluid and the structure subproblem.
- Semi-discretize the subproblems (in time) using the Backward Euler scheme.

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- Define approximate solutions, show its uniform boundedness and extract weak and weak* converging subsequences.
- Pass to the limit to see that the limiting functions satisfy the weak form of Problem 1.

Regularity of the weak solution

Formal energy estimates show that taking $(\mathbf{u}, \partial_t \boldsymbol{\eta}, \partial_t \mathbf{d}, \partial_t \mathbf{w})$ as a test function in the full, coupled problem, leads to the following regularity of the solution:

$$\begin{aligned}
&\mathbf{u} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V_F),\\
&\boldsymbol{\eta} \in W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;V_K),\\
&(\mathbf{d},\mathbf{w}) \in W^{1,\infty}(0,T;L^2(\mathcal{N})) \cap L^\infty(0,T;H^1(\mathcal{N})).
\end{aligned}$$

Time regularity - motivation

- One could take $(\partial_t \mathbf{u}, \partial_{tt}\boldsymbol{\eta}, \partial_{tt}\mathbf{d}, \partial_{tt}\mathbf{w})$ as a test function.
- The problem that appears is that we do not get the "right sign" in front of the elastic terms in structure equation.
- This is due to parabolic-hyperbolic-hyperbolic nature of the coupling between the fluid and composite structure.
- Taking $(\partial_{tt} \mathbf{u}, \partial_{ttt}\boldsymbol{\eta}, \partial_{ttt}\mathbf{d}, \partial_{ttt}\mathbf{w})$ solves this mismatch!

We define the time difference quotients in the following way:

$$D^{\Delta t}\mathbf{u}(t,\mathbf{x}) = \frac{\mathbf{u}(t+\Delta t,\mathbf{x}) - \mathbf{u}(t,\mathbf{x})}{\Delta t},$$

and define the test functions for our fluid-composite structure interaction problem as follows:

$$\begin{aligned} \boldsymbol{v} = -D^{-\Delta t}(D^{\Delta t}\mathbf{u}), \quad \boldsymbol{\psi} = -D^{-\Delta t}(D^{\Delta t}\partial_t\boldsymbol{\eta}), \\ \boldsymbol{\xi} = -D^{-\Delta t}(D^{\Delta t}\partial_t\mathbf{d}), \quad \boldsymbol{\zeta} = -D^{-\Delta t}(D^{\Delta t}\partial_t\mathbf{w}), \end{aligned} \tag{11}$$

Time regularity - estimates

The weak solution $(\mathbf{u}, \boldsymbol{\eta}, \mathbf{d}, \mathbf{w})$ of Problem 1 belongs to the following function spaces:

$$\begin{aligned}
&\mathbf{u} \in W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;V_F),\\
&\boldsymbol{\eta} \in W^{2,\infty}(0,T;L^2(R;\omega)) \cap W^{1,\infty}(0,T;V_K),\\
&(\mathbf{d},\mathbf{w}) \in W^{2,\infty}(0,T;L^2(\mathcal{N})) \cap W^{1,\infty}(0,T;V_S)
\end{aligned}$$

provided that initial data satisfy:

$$\mathbf{u}_0 \in H^2(\Omega),\ \boldsymbol{\eta}_0 \in V_K,\ \mathbf{v}_0 \in V_K,\ (\mathbf{d}_0, \mathbf{w}_0) \in V_S,\ (\mathbf{k}_0, \mathbf{z}_0) \in V_S$$

together with the compatibility conditions:

$$\nabla \cdot \mathbf{u}_0 = 0,\ (\mathbf{u}_0|_\Gamma \circ \boldsymbol{\varphi}) \cdot \mathbf{e}_r = (\mathbf{v}_0)_r,\ \mathbf{u}_0|_{\Gamma_{in/out}} \times \mathbf{e}_z = 0,\ \boldsymbol{\eta}_0 \circ \boldsymbol{\pi} = \mathbf{d}_0.$$

For the inlet and outlet pressure we demand $P_{in/out} \in H^1_{loc}(0,\infty)$.

Space regularity - formal estimates

One could naively take $(-\Delta\mathbf{u}, -\Delta\partial_t\boldsymbol{\eta}, -\Delta\partial_t\mathbf{d}, -\Delta\partial_t\mathbf{w})$ as a test function, where

$$\begin{aligned}\Delta\mathbf{u}(z,r,\theta) &= (\Delta u_z(z,r,\theta), \Delta u_r(z,r,\theta), \Delta u_\theta(z,r,\theta)) \\ &= (\partial_{zz}u_z + \partial_{rr}u_z + \partial_{\theta\theta}u_z, \partial_{zz}u_r + \partial_{rr}u_r + \partial_{\theta\theta}u_r, \\ &\quad \partial_{zz}u_\theta + \partial_{rr}u_\theta + \partial_{\theta\theta}u_\theta)\end{aligned}$$

and

$$\begin{aligned}\Delta\partial_t\boldsymbol{\eta}(z,\theta) &= (\Delta\partial_t\eta_z(z,\theta), \Delta\partial_t\eta_r(z,\theta), \Delta\partial_t\eta_\theta(z,\theta)) \\ &= (\partial_{zz}\partial_t\eta_z + \partial_{\theta\theta}\partial_t\eta_z, \partial_{zz}\partial_t\eta_r + \partial_{\theta\theta}\partial_t\eta_r, \partial_{zz}\partial_t\eta_\theta + \partial_{\theta\theta}\partial_t\eta_\theta).\end{aligned}$$

The problem that we encounter here is non-compatibility of the test functions, i.e. $\Delta\mathbf{u} \neq \Delta\partial_t\boldsymbol{\eta}$ on Γ.

Fluid interior regularity

- For the fluid test function we take $-\chi\Delta\mathbf{u}$, where χ is a smooth cut-off function which has support in the interior of the fluid domain.
- For the shell + mesh part we take zero test functions.
- The fluid test function is not divergence-free!
- One obtains an additional fluid interior regularity

$$\mathbf{u} \in L^\infty(0,T;H^1(\Omega_0)) \text{ and } \mathbf{u} \in L^2(0,T;H^2(\Omega_0)),$$

where $\Omega_0 \subset\subset \Omega$.

Shell interior regularity

- We now exclude the mesh from calculations.
- Take

$$\boldsymbol{v} = -\tilde{\chi}\Delta\mathbf{u} \text{ and } \boldsymbol{\psi} = -\chi\Delta\partial_t\boldsymbol{\eta}$$

  as a test function for the fluid ans shell equations, respectively.
- As we already noticed these two test functions are non-compatible, so we have to take slightly modified test function for the fluid part, namely:

$$\boldsymbol{v} = \tilde{\chi}(-\partial_{zz}u_{zz} - \partial_{\theta\theta}u_{zz}, -\partial_{zz}u_{rr} - \partial_{\theta\theta}u_{rr}, -\partial_{zz}u_{\theta\theta} - \partial_{\theta\theta}u_{\theta\theta}).$$

- For the fluid velocity, we obtain an additional regularity in z-direction and in θ-direction.
- An additional regularity of the fluid velocity in radial direction is obtained by using the Stokes equation.
- For the shell displacement, an additional regularity is obtained up to the boundary.

Mesh interior regularity

- In this step we calculate mesh interior regularity (by excluding the mesh vertices).
- Again we have to multiply the test functions with appropriate smooth cut-off functions.
- For the mesh, we take the following test functions

$$(-\Delta\partial_t \mathbf{d}_i, -\Delta\partial_t \mathbf{w}_i) = (-\partial_{ss}\partial_t \mathbf{d}_i, -\partial_{ss}\partial_t \mathbf{w}_i).$$

- For the fluid and the shell, we take

$$-\partial_{ss}\mathbf{u} \text{ and } -\partial_{ss}\partial_t \boldsymbol{\eta}.$$

- We obtain an additional fluid velocity and shell displacement regularity in s-direction.
- For the mesh, we obtain an additional regularity up to mesh vertices.