The two-variable Bollobás–Riordan polynomial of a connected even delta-matroid is irreducible

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21/6/21
The Tutte polynomial

The Tutte polynomial of a graph $G$ is given by

$$T(G; x, y) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)} (y - 1)^{|A| - r(A)},$$

where $r(A) = |V| - k(G|A)$, the number of edges in the largest forest of $G|A$. 
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Equivalently, $T(G; x, y) = 1$ if $G$ has no edges, and for each edge $e$,

$$T(G; x, y) = \begin{cases} 
  xT(G/e; x, y) & \text{if } e \text{ is a bridge}, \\
  yT(G \setminus e; x, y) & \text{if } e \text{ is a loop}, \\
  T(G/e; x, y) + T(G \setminus e; x, y) & \text{otherwise}.
\end{cases}$$
Irreducibility of $T$

**Theorem (Merino, de Mier, Noy (2001))**

$T(G; x, y)$ is irreducible in $\mathbb{C}[x, y]$ if and only if $G$ is 2-connected.

(This is also true for matroids.)
Key facts used in the proof

Write $T(G; x, y) = \sum_{i,j} b_{i,j} x^i y^j$. (We have $b_{i,j} \geq 0$.)

a. Brylawski's affine identities. For example,
   1. if $G$ has at least one edge then $b_{0,0} = 0$;
   2. if $G$ has at least two edges then $b_{1,0} = b_{0,1}$.

b. If $G$ has at least 2 edges, then $b_{1,0} \neq 0$ if and only if $G$ is 2-connected.

c. $T(G; x, y)$ has degree $r(E)$ in $x$ and if $G$ is loopless then $b_{r(E),0} = 1$ and otherwise $b_{r(E),i} = 0$.

d. $T(G; x, y)$ has degree $|E| - r(E)$ in $y$ and if $G$ is bridgeless then $b_{0,|E| - r(E)} = 1$ and otherwise $b_{i,|E| - r(E)} = 0$. 

The ribbon graph polynomial

For an orientable ribbon graph $G$ and set $A$ of its edges, let $g(A)$ denote the genus of the subgraph $G|A$.

Let $\sigma(A) = r(A) + g(A)$.
The ribbon graph polynomial

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Let \( \sigma(A) = r(A) + g(A) \).

We define the ribbon graph polynomial by

\[
R(G; x, y) = \sum_{A \subseteq E} (x - 1)^{\sigma(E) - \sigma(A)} (y - 1)^{|A| - \sigma(A)}.
\]
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$$R(G; x, y) = \sum_{A \subseteq E} (x - 1)^{\sigma(E) - \sigma(A)} (y - 1)^{|A| - \sigma(A)}.$$

We have

$$R(G; x, y) = (x - 1)^{g(G)} BR(G, x, y - 1, 1/\sqrt{(x - 1)(y - 1)}),$$

where $BR(G)$ is the Bollobás–Riordan polynomial of $G$. 
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If $G$ is a plane graph, then $R(G) = T(G)$. 
An example
An example

In this graph $\sigma(E) = 1$.

\[ R(G) = (x - 1) + 3(x - 1)(y - 1) + (x - 1)(y - 1)^2 + 2(y - 1) + (y - 1)^2 \]

\[ = xy^2 + xy - x - y, \]

which is irreducible.
An example

A ribbon graph is not 2-connected if its edges can be partitioned into sets $A$ and $B$, so that each circuit lies in either $A$ or $B$. 
A ribbon graph is not 2-connected if its edges can be partitioned into sets $A$ and $B$, so that each circuit lies in either $A$ or $B$ and no circuit in $A$ is interlaced with a circuit in $B$. 
An example

A ribbon graph is not **2-connected** if its edges can be partitioned into sets $A$ and $B$, so that each circuit lies in either $A$ or $B$ and no circuit in $A$ is interlaced with a circuit in $B$.

A loop is **trivial** if it is not **interlaced** with any other circuit.
Delete–Contract

If $G$ has no edges than $R(G; x, y) = 1$ and for each edge $e$,

- If $e$ is not a loop in either $G$ or $G^*$, then
  \[ R(G; x, y) = R(G \setminus e; x, y) + R(G/e; x, y). \]

- If $e$ is a loop in $G$ but not in $G^*$, then
  \[ R(G; x, y) = (x - 1)R(G \setminus e; x, y) + R(G/e; x, y). \]

- If $e$ is a loop in $G^*$ but not in $G$, then
  \[ R(G; x, y) = R(G \setminus e; x, y) + (y - 1)R(G/e; x, y). \]

- If $e$ is a loop in both $G$ and $G^*$, then
  \[ R(G; x, y) = (x - 1)R(G \setminus e; x, y) + (y - 1)R(G/e; x, y). \]
Key facts
Write $R(G; x, y) = \sum_{i,j} r_{i,j} x^i y^j$. (We no longer have $r_{i,j} \geq 0$.)

a. Brylawski’s affine identities.

Gordon (2015) showed that Brylawski’s affine identities hold extremely generally.

b. If $G$ has at least 2 edges, then $b_{1,0} \neq 0$ if and only if $G$ is 2-connected.

If $G$ has at least 2 edges, then $r_{1,0} \neq 0$ if and only if $G$ is 2-connected. This follows from a result of Bouchet (2001), which implies that if $G$ is 2-connected then at least one of $G \setminus e$ and $G / e$ is 2-connected.

c. $T(G; x, y)$ has degree $r(E)$ in $x$ and if $G$ is loopless then $b_{r(E),0} = 1$ and otherwise $b_{r(E),i} = 0$.

If $i > \sigma(E)$, then $r_{i,j} = 0$. Moreover

$$\sum_j r_{\sigma(E),j} = 1.$$
Main theorem

**Theorem**

If $G$ is an orientable ribbon graph, then $R(G; x, y)$ is irreducible if and only if $G$ is 2-connected.

(This extends to even delta-matroids.)
Thank you for listening.