Dual incidences and t-designs in elementary abelian groups

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Definition 1.2 *Let* $v = |\mathcal{H}| = \alpha_0\lambda$, $E_q[E_{q^n}] = \sum_{i=1}^{\left[n \atop 1\right]_q} \langle g_i \rangle$ *and* $\mathcal{H} = H_1 + \cdots + H_v$. *A matrix* $A = (A_{ij})_{\left[n \atop 1\right]_q \times \alpha_0\lambda}$*, given by*

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$$BB^t = \lambda(\alpha_0 - \beta)I + \beta\lambda J,$$

where $\beta = \dfrac{\begin{bmatrix} n-2 \\ k \end{bmatrix}_q}{\begin{bmatrix} n-t \\ k-t \end{bmatrix}_q}$.

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Theorem 1.8 *Matrices A and B satisfy $A = J - \frac{1}{q^{n-k-1}}CB$, where $C = (C_{ij})_{\left[\begin{smallmatrix} n \\ 1 \end{smallmatrix}\right]_q \times \left[\begin{smallmatrix} n \\ 1 \end{smallmatrix}\right]_q}$ is given by*

$$C_{ij} = \begin{cases} 1, & \textit{if } M_j \cap \langle g_i \rangle = 1 \\ 0, & \textit{otherwise.} \end{cases}$$

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Corollary 1.10 *For matrices* A, B, C, D *following holds:*

1. $A = J - \frac{1}{q^{n-k-1}} CJ + \frac{1}{q^{n-2}} CDA$
2. $B = J - \frac{1}{q^{k-1}} DJ + \frac{1}{q^{n-2}} DCB.$

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