Framed and Biframed Knotoids

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Definition

A knot is a smooth embedding $S^1 \hookrightarrow S^3$.

We consider such embeddings up to *ambient isotopy of* S^3:

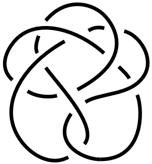


Figure: A planar projection of a knot.

A *knot diagram* is a immersion $S^1 \hookrightarrow S^2$, all whose crossings are transversal and endowed with crossing data.

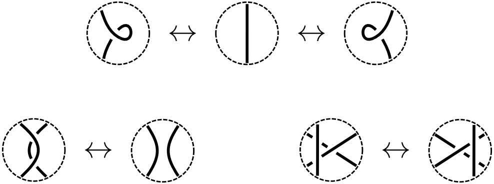
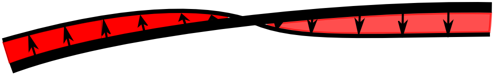


Figure: The Reidemeister moves $R1, R2, R3$.

Two knot diagrams are *equivalent* if they can be related by a sequence of isotopies and *Reidemeister relations*.

A *framed* knot is a knot with a transversal everywhere nonzero vector field. The associated element of $\pi_1(SO(2)) \cong \mathbb{Z}$ is the associated *framing integer*.

Equivalently, a framed knot is a knotted ribbon: an embedding $S^1 \times [0,1] \hookrightarrow S^3$.



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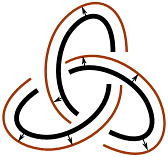
A knot diagram induces a canonical *blackboard framing* on the corrsponding knot, equal to its *writhe*.



$R2, R3$ preserve the writhe, but $R1$ changes it by ± 1. Instead framed knots correspond to knot diagrams up to $R1', R2, R3$.

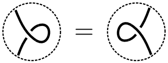


Figure: The weakened first Reidemeister move $R1'$.

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Definition

A *knotoid diagram* in S^2 is a knot diagram with open ends, considered up to isotopy and the Reidemeister moves; *away from the end-points*.

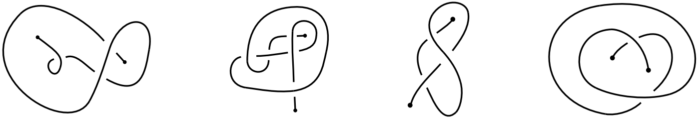

Figure: Examples of knotoids in the plane.

Definition

A θ-curve is a graph with vertices $\{v_0, v_1\}$ and edges $\{e_-, e_0, e_+\}$, embedded in S^3. They are considered up to label-preserving ambient isotopy. A θ-curve is *simple* if the embedding of $e_- \cup e_+$ is the unknot.

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Theorem

(Turaev, 2012): *There is a bijection between simple θ-curves up to equivalence in S^3 and knotoids in S^2.*

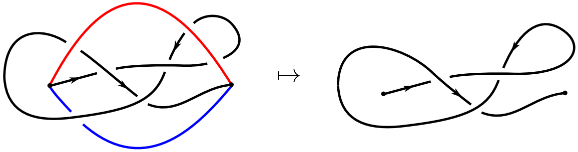

Figure: The knotoid-θ-curve correspondence.

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Definition

A *framed knotoid diagram* is a knotoid diagram, considered up to the same moves as knotoid diagrams, except for $R1$ which is replaced by $R1'$.

The *framing* of a framed knotoid diagram is given by its writhe.

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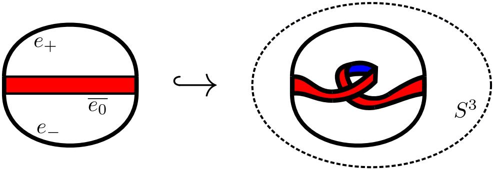
Framed θ-curves:



We consider *framed simple θ-curves*: embeddings in S^3 s.t. $e_+ \cup e_-$ is the unknot, up to label-preserving ambient isotopy.

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Theorem

(Moltmaker, 2021): *There is a bijection between simple framed θ-curves in S^3 and framed knotoids in S^2.*

Sketch Proof.

By Turaev's result for knotoids, it suffices to note the following bijections:

$$\begin{aligned}\{\mathrm{FKD}\} &\xleftrightarrow{\textit{writhe}} \{\mathrm{KD}\} \times \mathbb{Z} \\ &\xleftrightarrow{\textit{Turaev}} \{TC\} \times \mathbb{Z} \\ &\xleftrightarrow{\textit{framing}} \{FTC\}.\end{aligned}$$

□

Problem

Can we adjust our definitions to model half-twists?

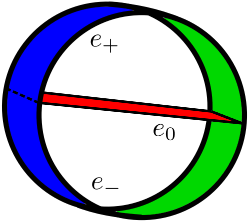
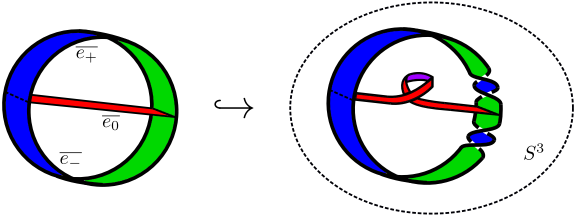


Figure: The model biframed θ-curve.

We'll consider *simple biframed θ-curves*, and their diagrams.

Biframed Theta-curves

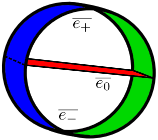
We consider embeddings such that $\overline{e}_+ \cup \overline{e}_-$ is the unframed unknot.



This will allow us to define:

- quantum invariants,
- half-twists (in some sense).

Consider the following embedded bi-framed θ-curves:



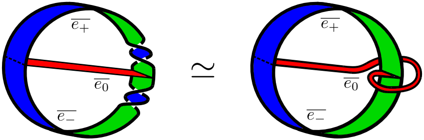
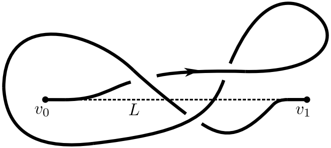


Figure: Inequivalent bi-framed θ-curves.

As simple θ-curves, they *would* be equivalent. This feature is the *coframing* of a biframed knotoid.

Fix $v_0, v_1 \in S^2$. Then biframed knotoids correspond to FKD's from v_0 to v_1 such that:

- The tangents at v_0, v_1 are equal and parallel to $L = \overrightarrow{v_0 v_1}$
- Isotopies fix a neighbourhood of the end-points



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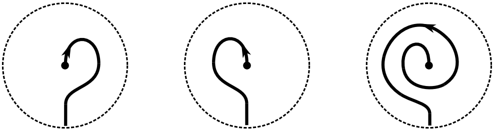
Question

How is the coframing represented?

The following have different coframing:



We define the coframing by:

$$C = W_0 - W_1$$

where $W_i = [\text{turning no. around } v_i]$.

Quantum Invariants

Construction for (Framed) Knots

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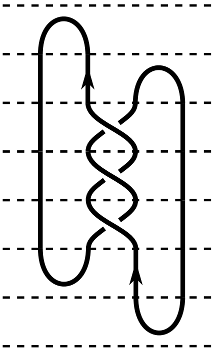
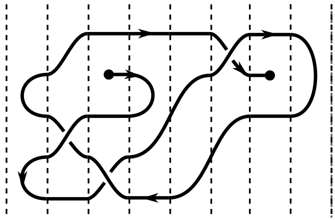


$\Longrightarrow$

$$
\begin{array}{c}
\mathbb{C} \\
\big\downarrow \ \mathrm{coev}_V \\
V^* \otimes V \\
\big\downarrow \ \mathrm{id}_V \otimes \mathrm{id}_{V^*} \otimes \overline{\mathrm{coev}}_V \\
V^* \otimes V \otimes V \otimes V^* \\
\big\downarrow \ \mathrm{id}_V \otimes \Psi_{V^*,V} \otimes \mathrm{id}_V \\
V^* \otimes V \otimes V \otimes V^* \\
\big\downarrow \ \mathrm{id}_V \otimes \Psi_{V,V^*} \otimes \mathrm{id}_V \\
V^* \otimes V \otimes V \otimes V^* \\
\big\downarrow \ \mathrm{id}_V \otimes \Psi_{V^*,V} \otimes \mathrm{id}_V \\
V^* \otimes V \otimes V \otimes V^* \\
\big\downarrow \ \overline{\mathrm{ev}}_V \otimes \mathrm{id}_{V^*} \otimes \mathrm{id}_V \\
V \otimes V^* \\
\big\downarrow \ \mathrm{ev}_V \\
\mathbb{C}
\end{array}
$$

Figure: Construction of a quantum knot invariant.

We *want* to make a Morse division:



But we can swivel at the endpoints!

$\implies$ #(ev's) and #(coev's) not well-defined

$\implies$ obstruction to constructing quantum invariants

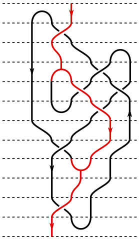
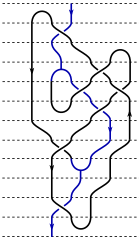
Quantum Invariants

For Biframed Knotoids

Solution:

Use biframed knotoid diagrams!

One option (Moltmaker, 2021): work with a bialgebra object in a braided category, and consider the morphisms given by the θ-curve:

$\varphi_+(K) = \varphi\Bigg($  $\Bigg)$ $\varphi_-(K) = \varphi\Bigg($  $\Bigg)$

Then $Q(K) = (\varphi_+(K), \varphi_-(K))$ is a biframed knotoid invariant.

References

- Elhamdadi, M., Hajij, M., & Istvan, K. (2020). Framed Knots. The Mathematical Intelligencer, 42(4), 7-22.
- Turaev, V. (2012). Knotoids. Osaka Journal of Mathematics, 49(1), 195-223.
- Gügümcü, N., & Kauffman, L. H. (2021). Quantum invariants of knotoids. arXiv pre-print 2102.12745

Thank you very much for you attention.