Taxonomy of Three-Qubit Doilies

Metod Saniga

Astronomical Institute
Slovak Academy of Sciences
SK-05960 Tatranská Lomnica
SLOVAKIA

(Joint work with **Henri de Boutray** (FEMTO-ST, UBFC, Besançon, France),
Frédéric Holweck (ICB/UTBM, UBFC, Belfort, France)
and **Alain Giorgetti** (FEMTO-ST, UBFC, Besançon, France))

8th European Congress of Mathematics
June 20 – 26, Portorož, Slovenia

Introduction

Some fifteen years ago it was discovered that there exists a deep connection between the structure of the N-qubit Pauli group and that of the binary symplectic polar space of rank N, $W(2N-1,2)$.

This connection has been, for example, used to get a deeper insight into:

- the nature of observable-based proofs of quantum contextuality ($\mathcal{Q}^+(3,2)$'s of $W(3,2)$ and Fano pentads of $W(5,2)$),
- the properties of certain black-hole entropy formulas and the so-called black-hole/qubit correspondence ($\mathcal{Q}^-(5,2)$ and $H(2)$ of $W(5,2)$),
- the finite-geometric underpinning of form theories of gravity (the 'magic' Veldkamp line of $W(5,2)$ and $W(7,2)$), and
- an intriguing finite-geometric toy model of space-time (the Klein quadric of $W(5,2)$).

Introduction

It is obvious that revealing finer traits of the structure of binary symplectic polar spaces of small rank can be vital for further physical applications of these spaces.

Having this in view, we will focus on sets of $W(3, 2)$'s (doilies) located in $W(5, 2)$, providing their comprehensive observable-based taxonomy.

Background: Polar Spaces

Given a d-dimensional projective space $\mathrm{PG}(d, 2)$ over $\mathrm{GF}(2)$, a *polar space* $\mathcal{P}$ in this projective space consists of the projective subspaces that are *totally isotropic/singular* with respect to a given non-singular bilinear form; $\mathrm{PG}(d, 2)$ is called the *ambient projective space* of $\mathcal{P}$.

A projective subspace of maximal dimension in $\mathcal{P}$ is called a *generator*; all generators have the same (projective) dimension $r - 1$.
One calls r the *rank* of the polar space.

Polar spaces of relevance for us are of three types:

- symplectic,
- hyperbolic and
- elliptic.

Background: Symplectic Polar Space

The *symplectic* polar space $W(2N-1,2)$, $N \geq 1$, consists of all the points of $\mathrm{PG}(2N-1,2)$, $\{(x_1, x_2, \ldots, x_{2N}) : x_j \in \{0,1\}, j \in \{1,2,\ldots,2N\}\} \backslash \{(0,0,\ldots,0)\}$, together with the totally isotropic subspaces with respect to the standard symplectic form

$$\sigma(x,y) = x_1 y_{N+1} - x_{N+1} y_1 + x_2 y_{N+2} - x_{N+2} y_2 + \cdots + x_N y_{2N} - x_{2N} y_N. \quad (1)$$

This space features

$$|W|_p = 4^N - 1 \quad (2)$$

points and

$$|W|_g = (2+1)(2^2+1)\cdots(2^N+1) \quad (3)$$

generators.

Background: Hyperbolic Orthogonal Polar Space

The *hyperbolic* orthogonal polar space $\mathcal{Q}^+(2N-1,2)$, $N \geq 1$, is formed by all the subspaces of $\mathrm{PG}(2N-1,2)$ that lie on a given non-singular hyperbolic quadric, with the standard equation

$$x_1 x_{N+1} + x_2 x_{N+2} \ldots + x_N x_{2N} = 0. \tag{4}$$

Each $\mathcal{Q}^+(2N-1,2)$ contains

$$|Q^+|_p = (2^{N-1}+1)(2^N-1) \tag{5}$$

points and there are

$$|W|_{Q^+} = |Q^+|_p + 1 = (2^{N-1}+1)(2^N-1) + 1 \tag{6}$$

copies of them in $W(2N-1,2)$.

Background: Elliptic Orthogonal Polar Space

The *elliptic* orthogonal polar space $\mathcal{Q}^-(2N-1,2)$, $N \geq 2$, comprises all points and subspaces of PG$(2N-1,2)$ satisfying the standard equation

$$f(x_1, x_{N+1}) + x_2 x_{N+2} + \cdots + x_N x_{2N} = 0, \qquad (7)$$

where f is a homogeneous irreducible polynomial of degree 2 over GF(2). Each $\mathcal{Q}^-(2N-1,2)$ contains

$$|Q^-|_p = (2^{N-1} - 1)(2^N + 1) \qquad (8)$$

points and there are

$$|W|_{Q^-} = |Q^-|_p + 1 = (2^{N-1} - 1)(2^N + 1) + 1 \qquad (9)$$

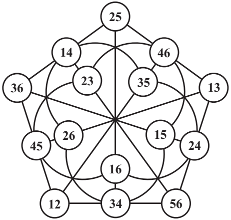
copies of them in $W(2N-1,2)$.

Background: $W(3,2)$, the Doily

The smallest non-trivial symplectic polar space is the $N = 2$ one, $W(3,2)$, often referred to as the *doily*.

It features 15 points and 15 lines (that are also its generators), with 3 points per line and 3 lines through a point.

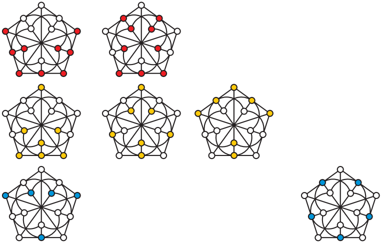
It is a self-dual 15_3-configuration and the only one out of 245 342 such configurations that is *triangle-free* (being isomorphic to the generalized quadrangle of order two (GQ(2,2))).



Background: $W(3,2)$, the Doily

This symplectic polar space features ten $\mathcal{Q}^+(3,2)$'s and six $\mathcal{Q}^-(3,2)$'s:

- $\mathcal{Q}^+(3,2)$ contains 9 points and 6 lines forming a 3×3 grid, so it is also called a grid;
- $\mathcal{Q}^-(3,2)$ features 5 pairwise non-collinear points partitioning the set of lines of the doily, hence it is an ovoid.



A triple of mutually non-collinear points of $W(3,2)$ is called a *triad*; a point collinear with all points of a triad is called a *center* of the triad. $W(3,2)$ contains 60 unicentric and 20 tricentric triads.

Background: N-qubit Observables

The N-qubit observables we will be dealing with belong to the set

$$\mathcal{S}_N = \{G_1 \otimes G_2 \otimes \cdots \otimes G_N : \ G_j \in \{I, X, Y, Z\},\ j \in \{1, \ldots, N\}\} \backslash \{\mathcal{I}_N\} \quad (10)$$

where $\mathcal{I}_N \equiv I_{(1)} \otimes I_{(2)} \otimes \ldots \otimes I_{(N)}$,

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (11)$$

are the Pauli matrices, I is the identity matrix and '$\otimes$' stands for the tensor product of matrices.

Background: N-qubit Observables

$\mathcal{S}_N$ features two kinds of observables:

- *symmetric* (those featuring an even number of Y's) and
- *skew-symmetric*;

the number of symmetric observables is $(2^{N-1}+1)(2^N-1)(= |Q^+|_p)$.

We shall also employ a finer classification where an observable having $N-1$, $N-2$, $N-3$, $\ldots$ I's will be, respectively, of type
A, B, C, $\ldots$.

(Also $G_1 \otimes G_2 \otimes \cdots \otimes G_N$ will be short-handed to $G_1 G_2 \cdots G_N$.)

Background: Observables $\leftrightarrow$ Points of $W(2N-1,2)$

For a particular value of N, the $4^N - 1$ elements of $\mathcal{S}_N$ can be bijectively identified with the same number of points of $W(2N-1,2)$ in such a way that

- the images of two commuting elements lie on <u>the same</u> line of this polar space, and
- *generators* of $W(2N-1,2)$ correspond to <u>maximal</u> sets of mutually commuting elements.

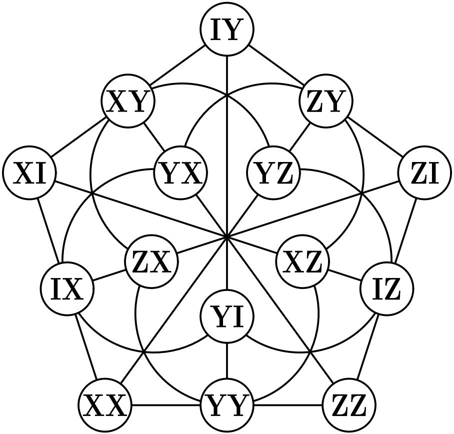
If we take the symplectic form defined by eq. (1), then this bijection acquires the form

$$G_j \leftrightarrow (x_j, x_{j+N}),\ j \in \{1,2,\ldots,N\}, \tag{12}$$

assuming that

$$I \leftrightarrow (0,0),\ X \leftrightarrow (0,1),\ Y \leftrightarrow (1,1),\ \text{and } Z \leftrightarrow (1,0). \tag{13}$$

Background: Two-Qubit Observables ↔ Points of $W(3,2)$



Background: Observables on Quadrics

Given an observable O,
– the set of *symmetric* observables *commuting* with O together with
– the set of *skew-symmetric* observables *not* commuting with O
lie on a certain non-degenerate quadric of $W(2N-1,2)$.

This quadric is hyperbolic (resp. elliptic) if O is symmetric (resp. skew-symmetric).

(We will express this property by making this associated observable explicit in a subscript, $\mathcal{Q}^{\pm}_{(O)}(2N-1,2)$.)

Background: Veldkamp Space

Given a point-line incidence geometry $\Gamma(P, L)$, a *geometric hyperplane* of $\Gamma(P, L)$ is a subset of its point set such that a line of the geometry is either *fully* contained in the subset or has with it just a *single* point in common.

The *Veldkamp* space $\mathcal{V}(\Gamma)$ of $\Gamma(P, L)$ is the space in which:

- a point is a geometric hyperplane of Γ and
- a line is the collection, denoted $H'H''$, of all geometric hyperplanes H of Γ such that $H' \cap H'' = H' \cap H = H'' \cap H$ or $H = H', H''$,

where H' and H'' are distinct points of $\mathcal{V}(\Gamma)$.

If $\Gamma(P, L)$ is a partial linear Gamma space with *three* points on a line, its Veldkamp lines are of the form $\{H', H'', \overline{H' \Delta H''}\}$ where $\overline{H' \Delta H''}$ is the complement of symmetric difference of H' and H'', i. e. they form a vector space over GF(2).

Background: Veldkamp Space of $W(2N-1,2)$

$\mathcal{V}(W(2N-1,2)) \cong \mathrm{PG}(2N,2)$.

Its points are

- hyperbolic quadrics,
- elliptic quadrics and
- perp-sets

of $W(2N-1,2)$.

Given a point x of $W(2N-1,2)$, the *perp-set* $\widehat{\mathcal{Q}}_{(x)}(2N-1,2)$ of x consists of all the points collinear with it,

$$\widehat{\mathcal{Q}}_{(x)}(2N-1,2) := \{y \in W(2N-1,2) \mid \sigma(x,y) = 0\}; \quad (14)$$

the point x being referred to as the *nucleus* of $\widehat{\mathcal{Q}}_{(x)}(2N-1,2)$.

Background: Veldkamp Space of $W(3,2)$ – points

We shall briefly recall basic properties of the Veldkamp space of the doily, $\mathcal{V}(W(3,2)) \simeq PG(4,2)$. The 31 points of $\mathcal{V}(W(3,2))$ comprise 10 grids, 15 perp-sets and 6 ovoids – as also illustrated in Figure 1.

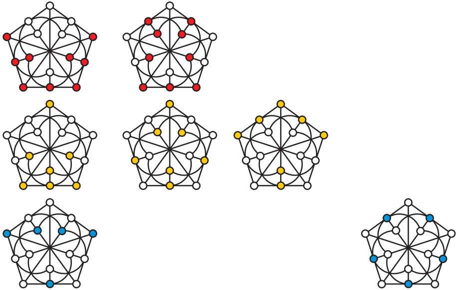


Figure: 1 – The three kinds of geometric hyperplanes of $W(3,2)$.

Background: Veldkamp Space of $W(3,2)$ – lines

The 155 lines of $\mathcal{V}(W(3,2))$ split into five distinct types as summarized in Table 1 and depicted in Figure 2.

Table: 1 – An overview of the properties of the five different types of lines of $\mathcal{V}(W(3,2))$ in terms of the *core* (i. e., the set of points common to all the three hyperplanes forming the line) and the types of geometric hyperplanes featured by a generic line of a given type. The last column gives the total number of lines per each type.

Type	Core	Perps	Ovoids	Grids	#
I	Two Secant Lines	1	0	2	45
II	Single Line	3	0	0	15
III	Tricentric Triad	3	0	0	20
IV	Unicentric Triad	1	1	1	60
V	Single Point	1	2	0	15

Background: Veldkamp Space of $W(3,2)$ – lines

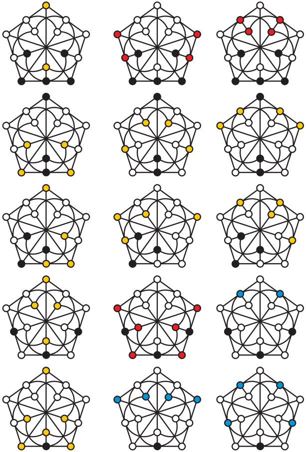


Figure: 2 – An illustrative portrayal of representatives (rows) of the five (numbered consecutively from top to bottom) different types of lines of $\mathcal{V}(W(3,2))$, each being uniquely determined by the properties of its core (black bullets).

Background: $W(2N-3,2)$'s in $W(2N-1,2)$

These are, in general, of two different kinds.

- A $W(2N-3,2)$ of the first kind, to be called *linear*, is isomorphic to the intersection of two perp-sets with non-collinear nuclei and their number in $W(2N-1,2)$ is
$$|W|_{W_l} = \frac{1}{3}4^{N-1}(4^N-1). \tag{15}$$
- A $W(2N-3,2)$ of the second kind, to be called *quadratic*, is isomorphic to the intersection of a hyperbolic quadric and an elliptic quadric and $W(2N-1,2)$ features
$$|W|_{W_q} = 4^{N-1}(4^N-1) \tag{16}$$
of them.

By way of example, in $W(3,2)$ a linear (resp. quadratic) $W(1,2)$ corresponds to a tricentric (resp. unicentric) triad.

Background: Positive/Negative subspaces

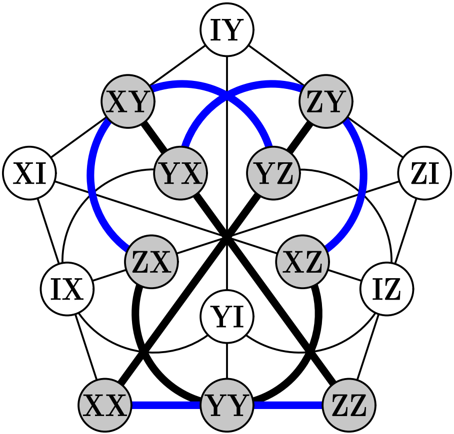
When referring to $W(2N-1,2)$ we will have in mind the $W(2N-1,2)$ whose points are labelled by N-qubit observables from the set $\mathcal{S}_N$ as expressed by eqs. (12) and (13).

A linear subspace of such $W(2N-1,2)$ will be called *positive* or *negative* according as the (ordinary) product of the observables located in it is $+\mathcal{I}_N$ or $-\mathcal{I}_N$, respectively.

Let us illustrate this point taking again $W(3,2)$, which features:

- 6 observables of type A (IX, XI, IY, YI, IZ, and ZI);
- 9 observables of type B (XX, XY, XZ, YX, YY, YZ, ZX, ZY and ZZ, these lying on a particular hyperbolic quadric, $\mathcal{Q}^{+}_{(YY)}(3,2)$); and
- 3 negative lines ($\{XX, YY, ZZ\}$, $\{XY, YZ, ZX\}$ and $\{XZ, YX, ZY\}$, these forming one system of generators of $\mathcal{Q}^{+}_{(YY)}(3,2)$).

Two-Qubit $W(3,2)$ in a Nutshell



$W(5,2)$: Observables and Negative Lines

The space $W(5,2)$ contains 63 points, 315 lines and 135 generators (Fano planes).

Among the 63 canonical three-qubit observables associated to the points, nine are of type A, 27 are type B and 27 are of type C.

Through an observable of type C there pass six negative lines, all being of type $C-C-B$; the total number of negative lines of this type thus is $\frac{27\times 6}{2}=81$.

Through an observable of type B there pass four negative lines. Of them, three are of the above-mentioned type and the fourth one is of type $B-B-B$; the total number of negative lines of the latter type is $\frac{27\times 1}{3}=9$.

As no negative line features an observable of type A, one finds that the $W(5,2)$ accommodates as many as $(81+9=)$ 90 negative lines.

$W(5,2)$: Positive and Negative Fano Planes

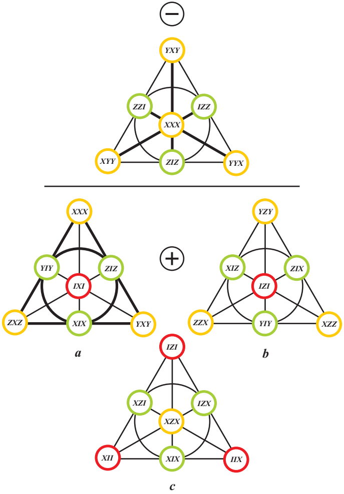


Figure: A representative of the family of 54 negative Fano planes (top) and those of three distinct types of positive Fano planes. Negative lines are shown in bold. An observable of type A, B and C is colored red, green and yellow, respectively.

$W(5,2)$: Doilies – Their Types and Numbers

$W(5,2)$ hosts:

- 336 linear doilies and
- 1008 quadratic ones.

A linear doily is located in a PG(3,2) of $W(5,2)$, this PG(3,2) being the polar space to a certain non-isotropic line of the ambient PG(5,2) induced by

$$\sigma(x,y) = x_1y_4 - x_4y_1 + x_2y_5 - x_5y_2 + x_3y_6 - x_6y_3.$$

$W(5,2)$: Doilies – Their Observable-Based Taxonomy

Table: 2 – Column one (T) shows the type, column two (C^-) the number of negative lines in a doily of the given type, columns three to five (O_A to O_C) indicate the number of observables of corresponding types, and columns six (D_l) and seven (D_q) yield, respectively, the number of linear and quadratic doilies of a given type. Type 9 splits further into two subtypes depending on whether the two observables of type A do (Type 9A, 162 members) or do not (Type 9B, 54 members) commute.

T	C^-	O_A	O_B	O_C	D_l	D_q
1	7	0	7	8	–	81
2	7	0	9	6	27	–
3	6	1	5	9	–	108
4	5	2	5	8	162	–
5	5	2	7	6	–	162
6	4	3	5	7	–	324
7	3	0	9	6	9	–
8	3	0	15	0	–	36
9	3	2	7	6	–	216
10	3	2	9	4	81	–
11	3	4	5	6	54	–
12	3	4	7	4	–	81
13	3	6	9	0	3	–

$W(5,2)$: Doilies and a Distinguished Quadric

The 27 observables of type B lie on the elliptic quadric $\mathcal{Q}^-_{(YYY)}(5,2)$ of $W(5,2)$; this special quadric, like any non-degenerate quadric, is a *geometric hyperplane* of $W(5,2)$.

As a doily is a *subgeometry* of $W(5,2)$, it

- either lies *fully* in $\mathcal{Q}^-_{(YYY)}(5,2)$ (Type 8),
- or shares with $\mathcal{Q}^-_{(YYY)}(5,2)$ a set of points that form

a *geometric hyperplane*; in particular,

- an ovoid (Types 3, 4, 6 and 11),
- a perp-set (Types 1, 5, 9 and 12) and
- a grid (Types 2, 7, 10 and 13).

One observes that no quadratic doily shares a grid with $\mathcal{Q}^-_{(YYY)}(5,2)$.

$W(5,2)$: Composite Doilies

Take the two-qubit doily.

Add formally to each observable, at the same position, the same mark from the set $\{X, Y, Z\}$.

Pick up a geometric hyperplane in this three-qubit labeled doily.

Replace by I the added mark in each observable that belongs to this geometric hyperplane.

One obviously gets a three-qubit doily.

Now, there are 31 geometric hyperplanes in the doily, 3 possibilities (X, Y, Z) to pick up a mark, and 3 possibilities (left, middle, right) where to insert the mark; so there are $31 \times 3 \times 3 = 279$ doilies created this way.

$W(5,2)$: Composite Doilies

In particular,

- out of the $15 \times 9 = 135$ doilies 'induced' by perp-sets, 81 are of Type 10 and 54 of Type 11;
- out of the $10 \times 9 = 90$ doilies 'generated' by grids, 81 are of Type 12 and 9 of Type 8; and, finally,
- the $6 \times 9 = 54$ doilies stemming from ovoids are all of the same Type $9B$.

So, if we look at the previous Table,

- all doilies of Types 1 to 7, 27 doilies of type 8 and all doilies of type $9A$ can be regarded as 'genuine' three-qubit guys, whereas
- 9 doilies of Type 8 that originate from grids (henceforth referred to as Type $8'$) and all doilies of types $9B$ to 13 can be viewed as composite, being 'built' from the two-qubit guy.

$W(5,2)$: Composite Doilies – 'Veldkamp' View

This stratification of three-qubit doilies can also be spotted in a different way. Take a representative doily of a particular type, for example that of Type 3 depicted in the next Figure, top.

From its three-qubit labels, keep first only the left mark (bottom left figure), then the middle mark (bottom middle figure) and, finally, the right mark (bottom right figure).

In each of these three 'residual' doilies it is easy to see that if you take the points featuring a given non-trivial mark (i.e., X, Y or Z) together with the points featuring I, these always form a geometric hyperplane, and the whole set forms a Veldkamp line of the doily where the points featuring I represent its core.

$W(5,2)$: Composite Doilies – ‘Veldkamp’ View

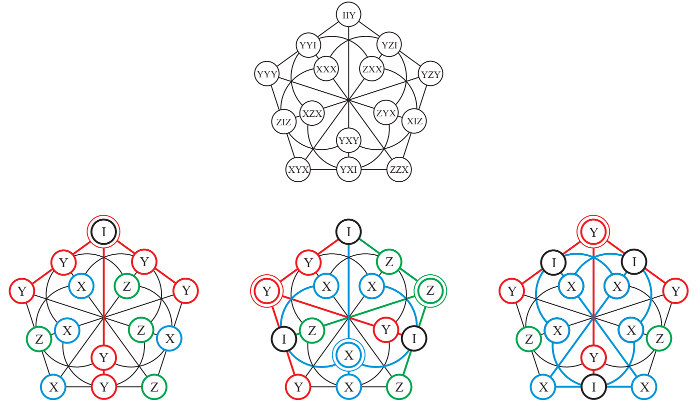


Figure: 3 – A formal decomposition of a three-qubit doily (top) into three ‘single-qubit residuals’ (bottom). In each doily of the bottom row, the three geometric hyperplanes forming a Veldkamp line are distinguished by different color, with their common points being drawn black; also, the nuclei of perp-sets are represented by double circles.

Employing Table 1 we readily see that this Veldkamp line is of type V (the core is a single point) for the left residual doily, type III (the core is a tricentric triad) for the middle doily and of type IV (the core is a unicentric triad) for the right one.

$W(5,2)$: Composite Doilies – 'Veldkamp' View

To account this way for all the 13 types of three-qubit doilies, we also need the concept of a *trivial* Veldkamp line of the doily, i. e. a line consisting of a geometric hyperplane counted twice and the full doily (and which entails those doilies 'generated' by the two-qubit doily).

This classification is summarized in the next Table. Here, columns two to six give the number of ordinary Veldkamp lines of a given type, columns seven to nine show the same for trivial Veldkamp lines and the last column corresponds to the degenerate case when all the points of a residual doily bear the label I.

Note that all doilies stemming from the two-qubit doily (i. e., Types $8'$ to 13) feature ordinary Veldkamp lines of the same type.

$W(5,2)$: Composite Doilies – ‘Veldkamp’ View

Table: 3 – A refined classification of doilies living in $W(5,2)$. We use the following abbreviations for the cores of Veldkamp lines: 2*cl* – two concurrent lines, *le* – line, *ttr* – tricentric triad, *utr* – unicentric triad, *pt* – point, *ov* – ovoid, *ps* – perp-set, *gr* – grid and *fl* stands for the full doily.

T	2cl	le	ttr	utr	pt	ov	ps	gr	fl
1	1	–	–	–	2	–	–	–	–
2	–	3	–	–	–	–	–	–	–
3	–	–	1	1	1	–	–	–	–
4	–	1	2	–	–	–	–	–	–
5	1	–	–	2	–	–	–	–	–
6	1	–	1	1	–	–	–	–	–
7	–	3	–	–	–	–	–	–	–
8	3	–	–	–	–	–	–	–	–
9A	1	–	–	2	–	–	–	–	–
8′	–	–	2	–	–	–	–	1	–
9B	–	–	2	–	–	1	–	–	–
10	–	–	2	–	–	–	1	–	–
11	–	–	2	–	–	–	1	–	–
12	–	–	2	–	–	–	–	1	–
13	–	–	2	–	–	–	–	–	1

$W(5,2)$: Four Doilies Sharing a Geometric Hyperplane

Using computer, we have also found out that given a doily and any geometric hyperplane in it, there are three other doilies having the same geometric hyperplane.

The Figure below serves as a visualisation of this fact when the common geometric hyperplane is an ovoid.

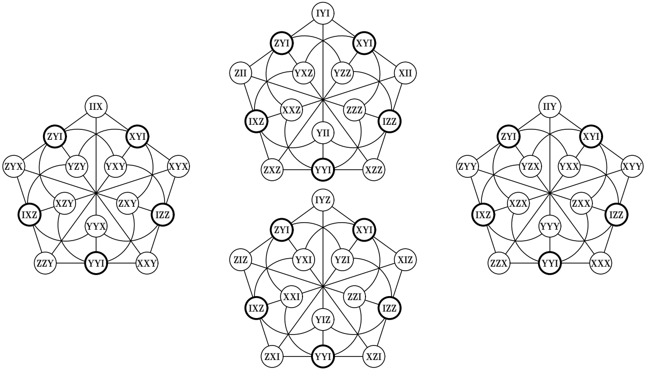


Figure: 4 – An illustration of the case when four different doilies share an ovoid (boldfaced). The top doily is of Type 11, the bottom one of Type 8, and both the left and right doilies are of Type 3.

$W(5,2)$: Four Doilies Sharing a Geometric Hyperplane

The four doilies sharing a geometric hyperplane, however, do not stand on the same footing, as can easily be discerned from our example depicted in Figure 4.

A point of the doily is collinear with three distinct points of an ovoid, the three points forming a *uni*centric triad.

Let us pick up such a triad, say $\{ZYI, XYI, YYI\}$, and look for its centers in each of the four doilies; these are IYI (top doily), IIX (left doily), IIY (right doily) and IYZ (bottom one).

We see that the last three observables are mutually anticommuting, whereas the first observable commutes with each of them.

This property is found to hold for each of $\binom{5}{3} = 10$ triads contained in an ovoid. Hence, the top doily of Figure 4 has indeed a different footing than the remaining three.

$W(5,2)$: Four Doilies Sharing a Geometric Hyperplane

A similar $3+1$ split-up is also observed in any quadruple of doilies having a *grid* in common because the three points of a grid collinear with a point of the doily also form a *uni*centric triad.

However, when the shared hyperplane is a *perp-set*, one gets a different, namely a $2+2$ split, because in this case the corresponding triple of points forms a *tri*centric triad.

$W(5,2)$: Four Doilies on a 'Planar' Tricentric Triad

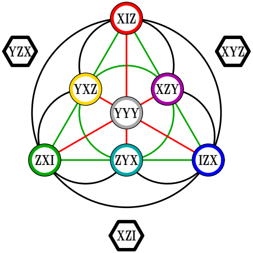
A tricentric triad of a linear resp. quadratic doily of $W(5,2)$ defines a line resp. plane in the ambient $\mathrm{PG}(5,2)$.

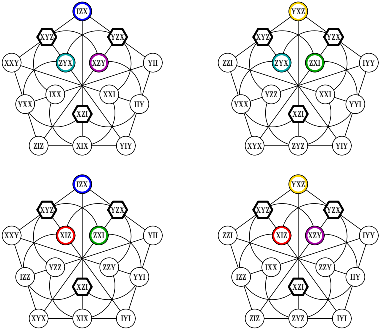
The latter type of a triad is found to be shared by four quadratic doilies.

- Given such a triad, there are 7 observables commuting with it, the corresponding 7 points laying in a Fano plane in the ambient $\mathrm{PG}(5,2)$.
- One of the 7 observables commutes with each of the remaining 6 ones, these 6 observables forming 3 commuting pairs.
- Out of the 6 guys one can form just 4 tricentric triads of which each is complementary to the triad we started from and thus defines with the latter a unique quadratic doily.

These properties are illustrated in the following figure:

$W(5,2)$: Four Doilies on a 'Planar' Tricentric Triad





An Outlook for Higher-Rank Taxonomy

When approaching this way subspaces of higher rank, it would be natural to include as parameters the number of negative linear subspaces of every viable dimension from 1 to $N-2$;

So, already in the case of $N=4$ we can add one more parameter, the number of negative planes a four-qubit $W(5,2)$ is endowed with.

As the three-qubit $W(5,2)$ features 54 negative planes, each composite four-qubit $W(5,2)$ must have the same number of negative planes.

An Outlook for Higher-Rank Taxonomy

Another interesting extension/variation of our taxonomy would be to take into account the number of negative lines passing through a point of the subspace.

Let us call this number the order of a point and for each subspace $W(2s-1,2)$ define the following string of parameters $[p_0, p_1, p_2, \ldots, p_{4^{s-1}-1}]$, where p_k, $0 \leq k \leq 4^{s-1}-1$, stands for the number of points of order k the subspace contains.

Applying this to three-qubit doilies ($s=2$), we find the following five patterns:

- $[0, 9, 6, 0]$ (Types 1 and 2),
- $[2, 9, 3, 1]$ (Type 3),
- $[5, 5, 5, 0]$ (Types 4 and 5),
- $[6, 6, 3, 0]$ (Type 6) and
- $[6, 9, 0, 0]$ (Types 7 to 13).

Reference

More details, including also the $N = 4$ case, can be found in:

- Saniga, M., de Boutray, H., Holweck, F., and Giorgetti, A.: 2021, Taxonomy of Polar Subspaces of Multi-Qubit Symplectic Polar Spaces of Small Rank, Journal of Physics A: Mathematical and Theoretical, submitted (arXiv:2105.03635).

THANK YOU FOR YOUR ATTENTION!