THE GEOMETRY OF RANDOM SPHERICAL EIGENFUNCTIONS

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Spherical eigenfunctions are defined as the solutions of the Helmoltz equation

$$\Delta_{\mathbb{S}^2} f_\ell + \lambda_\ell f_\ell = 0 \ , \ f_\ell : \mathbb{S}^2 \to \mathbb{R}, \ell = 1, 2, ...,$$

where $\Delta_{\mathbb{S}^2}$ is the spherical Laplacian and $\{-\lambda_\ell = -\ell(\ell+1)\}_{\ell=1,2,...}$ is the set of its eigenvalues. A random structure can be constructed easily by assuming that the eigenfunctions $\{f_\ell(.)\}$ follow a Gaussian isotropic random process on $\mathbb{S}^2$. More precisely, for each $x \in \mathbb{S}^2$, we take $\{f_\ell(x)\}$ to be a Gaussian random variable defined on a suitable probability space; without loss of generality, we assume these variables to have mean zero, unit variance, and covariance function given by

$$\mathbb{E}\left[f_\ell(x) f_\ell(y)\right] = P_\ell(\langle x, y\rangle) = P_\ell(\cos\theta_{xy}) \ ,$$

where $\theta_{xy} = d_{\mathbb{S}^2}(x, y)$ is the usual geodesic distance on the sphere, and $\{P_\ell(.)\}$ denotes the family of Legendre polynomials: this is the only covariance structure to ensure that the random eigenfunctions are isotropic, that is, invariant in law with respect to the action of the group of rotations $SO(3)$. Random spherical eigenfunctions, also known as random spherical harmonics, arise in a number of applications, especially in connection with Mathematical Physics: for instance, they represent the Fourier components of isotropic spherical random fields, whose analysis played an extremely important role in Cosmology over the last two decades; equally relevant is their role in Quantum Mechanics. Many of the results that we shall discuss can be easily generalized to eigenfunctions defined on higher-dimensional spheres or on different compact manifolds, but we will mainly stick to $\mathbb{S}^2$ for clarity and notational simplicity.

A lot of efforts have been spent in the last decade or two to characterize the geometry of the excursion sets for these random eigenfunctions, which are defined as

$$A_u(f_\ell; \mathbb{S}^2) := \left\{x \in \mathbb{S}^2 : f_\ell(x) \geq u\right\} \ , \ u \in \mathbb{R} \ .$$

A classical tool for the investigation of the excursion sets is provided by the so-called Lipschitz-Killing Curvatures (or equivalently, by Minkowski functionals), which in dimension 2 correspond to the Euler-Poincaré characteristic, the boundary length and the excursion area. A general expression for their expected values (covering more general Gaussian fields than random eigenfunctions) is given by the beautiful Gaussian Kinematic Formula, established in the first years of the XXI century by Taylor and Adler. Over the last decade, more refined characterizations have been obtained, including neat analytic expressions (in the high energy limit $\lambda_\ell \to \infty$) for the fluctuations around their expected values, the correlation among these different functionals, and the differences that are observed in some specific circumstances, such as the *nodal* case corresponding to $u = 0$. In this talk, we shall review these results, discussing in particular their connections with other areas of Probability, Geometry and Analysis, and presenting open issues for future research.

My own research in this area is largely the outcome of a long lasting collaboration with Igor Wigman; the collaboration has involved several other coauthors, including in particular Valentina Cammarota, Giovanni Peccati and Maurizia Rossi.