Birational algebraic geometry

Yuri Prokhorov

Birational geometry is the field of algebraic geometry that studies algebraic varieties up to birational equivalence, that is, modulo maps which are isomorphisms on Zariski open subsets. The easiest example of a birational map is the map $\mathbb{P}^2_{x,y} \dashrightarrow \mathbb{P}^1_{t_1} \times \mathbb{P}^1_{t_2}$ of projective plane to the product of two projective lines given by $(x, y) \longmapsto (x, y)$. This map is an isomorphism on the main affine chart. On the other hand, these two surfaces are not isomorphic biregularly because $\mathbb{P}^1 \times \mathbb{P}^1$ contains a lot pairwise disjoint curves but any two curves on the projective plane $\mathbb{P}^2$ have at least one common point.

Birational geometry in dimension one is quite simple: two smooth projective curves are birationally equivalent if and only if they are isomorphic. In higher dimension the situation becomes much more complicated. In fact, one can find infinitely many, even smooth and projective, elements in any birational equivalence class of dimension ≥ 2. The goal of birational geometry is to find some distinguished elements in each birational class and then classify such elements.

Below we list some topics in birational geometry which are close to author's field of research.

Mori theory. The minimal model program (MMP) is the most powerful method in the classification theory of algebraic varieties [Mor88]. This is a natural generalization of the classical theory of algebraic surfaces by Enriques and Castelnuovo. The core of the method appeared in the late 1970s and earlier 1980s in works of Reid, Mori, Kawamata, Shokurov and others. The main very rough idea of this program is to apply certain special transformations to an algebraic variety and obtain "simpler" model. At present, the existence of the minimal models is established in arbitrary dimension [BCHM10] but, even in dimension three, a knowledge of their complete birational characteristics is still lacking. The main problem here is that higher-dimensional Mori theory works in category varieties with certain mild singularities. In recent years the effective version of the three-dimensional MMP was developed essentially [MP14], [MP16]. See [MP19] for the most recent survey.

Classification of Fano varieties. The notion of Fano varieties, introduced by G. Fano and V. Iskovskikh. By definition these are projective varieties with ample anticanonical class. They appear naturally as one of the outputs of the MMP. Classification of smooth Fano varieties is known up to dimension three (see [IP99]). But the minimal model program has also led to the study of Fano varieties with singularities. The fundamental result in this direction is the boundedness (BAB) established recently [Bir16]. It is expected that in dimension three effective classification of Fano varieties with mild (e.g. terminal) singularities is possible. See e.g. [Pro10], [Pro05] for partial classificational results.

Rationality problems. An algebraic variety is said to be rational if it is birationally equivalent to the projective space $\mathbb{P}^n$. In dimension 1 and 2 there are rationality criterions. They are formulated in terms of differential forms. On the other hand, one cannot expect similar criterion in dimension ≥ 3. For example, all natural differential and topological birational invariants vanish for Fano varieties but among them there are a lot of non-rational ones (three-dimensional cubic, quartic etc). Thus the rationality problems in higher dimensions is much more delicate. The most challenging problem here is the rationality criterion of three-dimensional conic bundles [Pro18].

Cremona groups. The Cremona group of rank n is the group of birational self-maps $\mathrm{Bir}(\mathbb{P}^n)$ of the projective space $\mathbb{P}^n$ or, equivalently, the group of $\Bbbk$-automorphisms of $\Bbbk(x_1, \ldots, x_n)$, the field of rational functions in n independent variables. For $n \geq 2$, this group is huge and is very far from being linear algebraic. However, the system of its finite subgroups enjoys many features of finite subgroups in $\mathrm{GL}_n(\Bbbk)$. For example, Cremona groups $\mathrm{Bir}(\mathbb{P}^n)$ satisfy the Jordan property [Ser09], [PS16], [Bir16]. Similar behavior can be detected for the groups of birational self-maps of many other algebraic varieties [PS14].

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