NONCOMMUTATIVE RESOLUTIONS OF SINGULARITIES

Resolutions of singularities are a fundamental tool in algebraic geometry for dealing with singular spaces. The basic idea is to replace a singular space with a nonsingular one which closely resembles it. Hironaka [Hir64] proved that a resolution of singularities always exists (in characteristic 0). Curves and surface admit a unique minimal resolution of singularities, but in higher dimensions this is no longer true. However, the best behaved resolutions are so-called "crepant resolutions", i.e. those which preserve the bundle of volume forms. There is a general belief that all crepant resolutions should have the same (co)homological structure; more precisely, that their "derived categories" should be equivalent. This is the famous Bondal-Orlov conjecture [BO02], which serves as a substitute for the lack of a unique minimal resolution. In dimension three the conjecture was proved by Bridgeland [Bri02] and Kawamata [Kaw16] established it for toric varieties. In other cases the conjecture is still wide open. An important evidence for the conjecture is Batyrev's result [Bat00] showing that crepant resolutions have the same Hodge numbers.

An important recent development in algebraic geometry is the appearance of *noncommutative resolutions* [VdB04]. The idea, which first occurred in physics (e.g. [BL01, DGM97]), is similar to the idea of classical (commutative) resolutions. A space is by the geometry/algebra duality replaced by a ring, and thus one looks for a non-singular (possibly noncommutative) ring that replaces the original (singular) ring. Non-commutative resolutions equip the geometry with extra structure and sometimes provide a more transparent way to understand the initial singularity than do commutative resolutions.

With Michel Van den Bergh we constructed noncommutative (crepant) resolutions of quotient singularities for general reductive groups (which form an important and large class of singularities) using algebraic methods [ŠVdB17]. These extend the classical and celebrated constructions known only for quotient singularities for finite groups. Moreover, combining [HL15] with methods in [ŠVdB17] Halpern-Leistner and Sam [HLS16] deduce that various GIT quotients of certain linear representations of reductive groups are derived equivalent, which provides a significant progress on the Bondal-Orlov Conjecture.

This is an example of many recent advancements in "derived" algebraic geometry where it has become apparent that the category of (commutative) spaces is too small to admit certain natural constructions and that it has to be enlarged to allow noncommutative spaces (in a wide sense).

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