GENERALIZATIONS OF STEFFENSEN'S INEQUALITY BY INTERPOLATING POLYNOMIALS

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Abstract

In papers [3, 4, 5] we obtain new generalizations of Steffensen's inequality using two-point Abel-Gontscharoff polynomial, Hermite interpolating polynomials and Lidstone's polynomial. Here we present results obtained by one of the polynomials i.e. Lidstone's polynomial. Firstly, we give few valuable identities and then using these identities we obtain new generalizations of Steffensen's inequality for $(2n)-$convex functions. Further, using Čebyšev inequality we consider the bounds for the integrals in the perturbed versions of the previously described identities.
In [5], are also given new generalizations of Steffensen's inequality for $(2n+1)-$convex functions, as well as Grüss type inequalities for the integrals in the perturbed versions of the obtained identities.

Introduction

At the beginning we give known results that we use.

The well-known Steffensen's inequality states ([6]):

Theorem 1. Suppose that f is nonincreasing and g is integrable on $[a,b]$ with $0 \leq g \leq 1$ and $\lambda = \int_a^b g(t)dt$. Then we have

$$\int_{b-\lambda}^b f(t)dt \leq \int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt.$$

The inequalities are reversed for f nondecreasing.

In [7] Widder proved the following fundamental lemma:

Lemma 1. If $f \in C^{(2n)}([0,1])$, then

$$f(t) = \sum_{k=0}^{n-1}\left[f^{(2k)}(0)\Lambda_k(1-t) + f^{(2k)}(1)\Lambda_k(t)\right] + \int_0^1 G_n(t,s)f^{(2n)}(s)ds,$$

where G_n is Green's function defined by

$$G_1(t,s) = G(t,s) = \begin{cases} (t-1)s, & \text{if} \quad s < t, \\ (s-1)t, & \text{if} \quad t \leq s, \end{cases}$$

$$G_n(t,s) = \int_0^1 G_1(t,p)G_{n-1}(p,s)dp, \qquad n \geq 2$$

and Λ_n is the unique polynomial (Lidstone polynomial) of degree $2n+1$, $n \in \mathbb{N}$, defined on interval $[0,1]$ by

$$\Lambda_0(t) = t, \qquad \Lambda_n''(t) = \Lambda_{n-1}(t), \qquad \Lambda_n(0) = \Lambda_n(1), \quad n \geq 1.$$

In [2] Jakšetić and Pečarić generalized Steffensen's inequality for positive measures using following identities

$$\int_{[a,a+\lambda]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) = \int_{[a,a+\lambda]} f(t)(1-g(t))d\mu(t) - \int_{(a+\lambda,b]} f(t)g(t)d\mu(t)$$

and

$$\int_{[a,b]} f(t)g(t)d\mu(t) - \int_{(b-\lambda,b]} f(t)d\mu(t) = \int_{[a,b-\lambda]} f(t)g(t)d\mu(t) - \int_{(b-\lambda,b]} f(t)(1-g(t))d\mu(t).$$

The above identities for $d\mu(t) = p(t)dt$ will be the starting point for the generalizations of Steffensen's inequality.

In the sequel we give the theorem that is used to obtain some new bounds for the reminders in the new identities.

Let $f, h : [a,b] \to \mathbb{R}$ be Lebesgue integrable functions. We define the Čebyšev functional $T(f,h)$ by

$$T(f,h) := \frac{1}{b-a}\int_a^b f(t)h(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b h(t)dt.$$

In [1] Cerone and Dragomir proved the following theorem:

Theorem 2. Let $f : [a,b] \to \mathbb{R}$ be a Lebesgue integrable function and $h : [a,b] \to \mathbb{R}$ be an absolutely continuous function with $(\cdot - a)(b - \cdot)[h']^2 \in L_1[a,b]$. Then we have the inequality

$$|T(f,h)| \leq \frac{1}{\sqrt{2}}[T(f,f)]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left(\int_a^b (x-a)(b-x)[h'(x)]^2 dx\right)^{\frac{1}{2}}. \tag{1}$$

The constant $\frac{1}{\sqrt{2}}$ in (1) is the best possible.

References

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Generalizations of Steffensen's inequality by Lidstone's polynomial

Firstly, using Lidstone's interpolating polynomial we obtain new identities related to the Steffensen's inequality.

Theorem 3. Let $f : I \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous for some $n \geq 1$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$ and let the function $\mathcal{G}_1$ be defined by

$$\mathcal{G}_1(x) = \begin{cases} \int_a^x (1-g(t))p(t)dt, & x \in [a, a+\lambda], \\ \int_x^b g(t)p(t)dt, & x \in [a+\lambda, b]. \end{cases} \tag{2}$$

Then

$$\begin{aligned} &\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt + \sum_{k=0}^{n-1}(b-a)^{2k-1}\int_a^b \mathcal{G}_1(x)\left[f^{(2k)}(b)\,\Lambda_k'\left(\frac{x-a}{b-a}\right) - f^{(2k)}(a)\,\Lambda_k'\left(\frac{b-x}{b-a}\right)\right]dx \\ &= -(b-a)^{2n-1}\int_a^b\left(\int_a^b \mathcal{G}_1(x)\frac{dG_n}{dx}\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right)dx\right)f^{(2n)}(s)ds. \end{aligned} \tag{3}$$

Theorem 4. Let $f : I \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous for some $n \geq 1$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$ and let the function $\mathcal{G}_2$ be defined by

$$\mathcal{G}_2(x) = \begin{cases} \int_a^x g(t)p(t)dt, & x \in [a, b-\lambda], \\ \int_x^b (1-g(t))p(t)dt, & x \in [b-\lambda, b]. \end{cases} \tag{4}$$

Then

$$\begin{aligned} &\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt + \sum_{k=0}^{n-1}(b-a)^{2k-1}\int_a^b \mathcal{G}_2(x)\left[f^{(2k)}(b)\,\Lambda_k'\left(\frac{x-a}{b-a}\right) - f^{(2k)}(a)\,\Lambda_k'\left(\frac{b-x}{b-a}\right)\right]dx \\ &= -(b-a)^{2n-1}\int_a^b\left(\int_a^b \mathcal{G}_2(x)\frac{dG_n}{dx}\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right)dx\right)f^{(2n)}(s)ds \end{aligned} \tag{5}$$

Secondly, using obtained identities we give the generalization of the Steffensen's inequality for $(2n)$-convex functions.

Theorem 5. Let $f : I \to \mathbb{R}$ be such that $f^{(2n-1)}$ is absolutely continuous for some $n \geq 1$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let the functions $\mathcal{G}_1$,$\mathcal{G}_2$ be defined by (2) and (4) respectively.

(i) Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$. If f is $(2n)$-convex function and

$$\int_a^b \mathcal{G}_1(x)\frac{dG_n}{dx}\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right)dx \geq 0 \tag{6}$$

then

$$\int_a^b f(t)g(t)p(t)dt \geq \int_a^{a+\lambda} f(t)p(t)dt + \sum_{k=0}^{n-1}(b-a)^{2k-1}\int_a^b \mathcal{G}_1(x)\left[f^{(2k)}(b)\,\Lambda_k'\left(\frac{x-a}{b-a}\right) - f^{(2k)}(a)\,\Lambda_k'\left(\frac{b-x}{b-a}\right)\right]dx. \tag{7}$$

(ii) Let $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$. If f is $(2n)$-convex function and

$$\int_a^b \mathcal{G}_2(x)\frac{dG_n}{dx}\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right)dx \geq 0 \tag{8}$$

then

$$\int_a^b f(t)g(t)p(t)dt \leq \int_{b-\lambda}^b f(t)p(t)dt - \sum_{k=0}^{n-1}(b-a)^{2k-1}\int_a^b \mathcal{G}_2(x)\left[f^{(2k)}(b)\,\Lambda_k'\left(\frac{x-a}{b-a}\right) - f^{(2k)}(a)\,\Lambda_k'\left(\frac{b-x}{b-a}\right)\right]dx. \tag{9}$$

The reversed inequalities in (6) and (8) implies the reversed inequalities in (7) and (9) respectively.

Inequalities related to the bounds for the Čebyšev functional

We use Theorem 2 to obtain some new bounds for integrals on the left hand side in the perturbed version of the identities (3) and (5). Firstly, let us denote

$$\Omega_i(s) = \int_a^b \mathcal{G}_i(x)\frac{dG_n}{dx}\left(\frac{x-a}{b-a}, \frac{s-a}{b-a}\right)dx, \quad i = 1, 2 \tag{10}$$

Theorem 6. Let $f : [a,b] \to \mathbb{R}$ be such that $f \in C^{2n}([a,b])$ and $(\cdot - a)(b - \cdot)[f^{(2n+1)}]^2 \in L_1[a,b]$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$ and let the functions $\mathcal{G}_1$ and Ω_1 be defined by (2) and (10). Then

$$\begin{aligned} &\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt + \sum_{k=0}^{n-1}(b-a)^{2k-1}\int_a^b \mathcal{G}_1(x)\left[f^{(2k)}(b)\,\Lambda_k'\left(\frac{x-a}{b-a}\right) - f^{(2k)}(a)\,\Lambda_k'\left(\frac{b-x}{b-a}\right)\right]dx \\ &+ (b-a)^{2n-2}\left(f^{(2n-1)}(b) - f^{(2n-1)}(a)\right)\int_a^b \Omega_1(s)ds = S_n^1(f;a,b), \end{aligned}$$

where the remainder $S_n^1(f;a,b)$ satisfies the estimation $\left|S_n^1(f;a,b)\right| \leq \frac{(b-a)^{2n}}{\sqrt{2}}[T(\Omega_1,\Omega_1)]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left(\int_a^b (s-a)(b-s)[f^{(2n+1)}(s)]^2 ds\right)^{\frac{1}{2}}$.

Theorem 7. Let $f : [a,b] \to \mathbb{R}$ be such that $f \in C^{2n}([a,b])$ and $(\cdot - a)(b - \cdot)[f^{(2n+1)}]^2 \in L_1[a,b]$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$ and let the functions $\mathcal{G}_2$ and Ω_2 be defined by (4) and (10). Then

$$\begin{aligned} &\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt + \sum_{k=0}^{n-1}(b-a)^{2k-1}\int_a^b \mathcal{G}_2(x)\left[f^{(2k)}(b)\,\Lambda_k'\left(\frac{x-a}{b-a}\right) - f^{(2k)}(a)\,\Lambda_k'\left(\frac{b-x}{b-a}\right)\right]dx \\ &+ (b-a)^{2n-2}\left(f^{(2n-1)}(b) - f^{(2n-1)}(a)\right)\int_a^b \Omega_2(s)ds = S_n^2(f;a,b), \end{aligned}$$

where the remainder $S_n^2(f;a,b)$ satisfies the estimation $\left|S_n^2(f;a,b)\right| \leq \frac{(b-a)^{2n}}{\sqrt{2}}[T(\Omega_2,\Omega_2)]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left(\int_a^b (s-a)(b-s)[f^{(2n+1)}(s)]^2 ds\right)^{\frac{1}{2}}$.