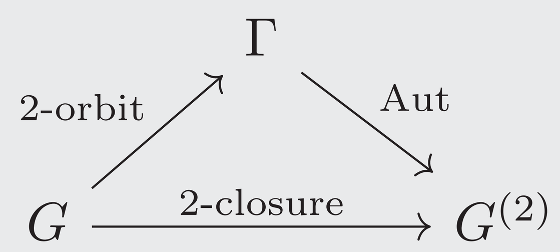
On 2-closures of rank 3 groups

Saveliy V. Skresanov
Sobolev Institute of Mathematics
skresan@math.nsc.ru

2-Closures and rank 3

Groups are often utilized in combinatorics to build highly symmetrical structures, one of the most prominent examples being the construction of the so-called rank 3 graphs. Recall that a permutation group $G \leq \mathrm{Sym}(\Omega)$ acts on the set of pairs $\Omega \times \Omega$ coordinatewise; an orbit of such action is called a 2-orbit. The largest permutation group on Ω having the same 2-orbits as G is called the **2-closure** of G and is denoted by $G^{(2)}$.

If the group has only three 2-orbits, then it is called a **rank 3 group**; a nondiagonal 2-orbit of such a group is called a **rank 3 graph**. A rank 3 graph Γ associated with the rank 3 group G is typically a strongly regular graph, and its full automorphism group is precisely the 2-closure of G:



Our contribution

Famously, the classification of rank 3 groups was completed in [1] (see also [2] for an overall overview).

In this work, building on that classification, we describe the 2-closures of all rank 3 groups or, in other words, full automorphism groups of rank 3 graphs. Groups are organized into several classes, highlighting the combinatorial structures preserved.

Main result

Let G be a rank 3 permutation group on a set Ω. Then besides a finite number of exceptions one of the following is true.

(i) G is imprimitive, i.e. it preserves a nontrivial decomposition $\Omega = \Delta \times X$. Then $G^{(2)} = \mathrm{Sym}(\Delta) \wr \mathrm{Sym}(X)$.

(ii) G is primitive and preserves a product decomposition $\Omega = \Delta^2$. Then $G^{(2)} = \mathrm{Sym}(\Delta) \uparrow \mathrm{Sym}(2)$.

(iii) G is primitive almost simple with socle L, i.e. $L \trianglelefteq G \leq \mathrm{Aut}(L)$. Then $G^{(2)} = N_{\mathrm{Sym}(\Omega)}(L)$, and $G^{(2)}$ is almost simple with socle L.

(iv) G is a primitive affine group which does not stabilize a product decomposition. Then $G^{(2)}$ is also an affine group. More precisely, there exist an integer $a \geq 1$ and a prime power q such that $G \leq \mathrm{A\Gamma L}_a(q)$, and exactly one of the following holds:

(a) $G \leq \mathrm{A\Gamma L}_1(q)$. Then $G^{(2)} \leq \mathrm{A\Gamma L}_1(q)$.

(b) $G \leq \mathrm{A\Gamma L}_{2m}(q)$ preserves the bilinear forms graph $H_q(2, m)$, $m \geq 3$. Then

$$G^{(2)} = \mathbb{F}_q^{2m} \rtimes ((\mathrm{GL}_2(q) \circ \mathrm{GL}_m(q)) \rtimes \mathrm{Aut}(\mathbb{F}_q)).$$

(c) $G \leq \mathrm{A\Gamma L}_{10}(q)$ preserves the alternating forms graph $A(5, q)$. Then

$$G^{(2)} = \mathbb{F}_q^{10} \rtimes ((\Gamma\mathrm{L}_5(q)/\{\pm 1\}) \times (\mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2)).$$

(d) $G \leq \mathrm{A\Gamma L}_{2m}(q)$ preserves the affine polar graph $\mathrm{VO}_{2m}^{\epsilon}(q)$, where $m \geq 2$, $\epsilon = \pm$. Then

$$G^{(2)} = \mathbb{F}_q^{2m} \rtimes \Gamma\mathrm{O}_{2m}^{\epsilon}(q).$$

(e) $G \leq \mathrm{A\Gamma L}_4(q)$ preserves the Suzuki-Tits ovoid graph $\mathrm{VSz}(q)$, $q = 2^{2e+1}$, $e \geq 1$. Then

$$G^{(2)} = \mathbb{F}_q^4 \rtimes ((\mathbb{F}_q^\times \times \mathrm{Sz}(q)) \rtimes \mathrm{Aut}(\mathbb{F}_q)).$$

(f) $G \leq \mathrm{A\Gamma L}_{16}(q)$ preserves the affine half spin graph $\mathrm{VD}_{5,5}(q)$. Then $G^{(2)} \leq \mathrm{A\Gamma L}_{16}(q)$ and

$$G^{(2)} = \mathbb{F}_q^{16} \rtimes ((\mathbb{F}_q^\times \circ \mathrm{Inndiag}(D_5(q))) \rtimes \mathrm{Aut}(\mathbb{F}_q)),$$

where $\mathrm{Inndiag}(D_5(q))$ is the overgroup of $D_5(q)$ in $\mathrm{Aut}(D_5(q))$, containing all diagonal automorphisms.

Group classes

$\Delta \times X$	G is **imprimitive**, i.e. G preserves a system of cliques of equal sizes. Here G acts on $\Omega = \Delta \times X$ by permuting blocks $\Delta \times \{x\}$, $x \in X$, each block corresponding to a clique. The group naturally lies in the imprimitive wreath product $\mathrm{Sym}(\Delta) \wr \mathrm{Sym}(X)$.
Δ^2	G preserves a **product decomposition** $\Omega = \Delta^2$, i.e. one of the 2-orbits of G is a Hamming graph $H(\lvert\Delta\rvert, 2)$ of diameter two. The group lies in the primitive wreath product $\mathrm{Sym}(\Delta) \uparrow \mathrm{Sym}(2)$.
$L \trianglelefteq G$	G is **almost simple**, i.e. $L \trianglelefteq G \leq \mathrm{Aut}(L)$, where L is nonabelian simple. Many graphs here, for example, triangular graph for $L = \mathrm{Alt}(n)$ and polar graphs for classical groups.
$\mathrm{A\Gamma L}_1(q)$	G lies in the **affine semilinear group** $\mathrm{A\Gamma L}_1(q)$, i.e. $\mathbb{F}_q^+ \leq G \leq \mathbb{F}_q^+ \rtimes (\mathbb{F}_q^\times \rtimes \mathrm{Aut}(\mathbb{F}_q))$. The graphs here are Paley, Peisert or Van Lint–Schrijver graphs.
$H_q(2, m)$	G preserves the **bilinear forms graph** $H_q(2, m)$, i.e. the vertex set Ω can be identified with the set of $2 \times m$ matrices over $\mathbb{F}_q$, and vertices $A, B \in \Omega$ are adjacent iff $\mathrm{rk}(A - B) = 1$.
$A(5, q)$	G preserves the **alternating forms graph** $A(5, q)$, i.e. the vertex set Ω can be identified with the set of 5×5 skew-symmetric matrices over $\mathbb{F}_q$, and $A, B \in \Omega$ are adjacent iff $\mathrm{rk}(A - B) = 2$.
$\mathrm{VO}_{2m}^{\epsilon}(q)$	G preserves the **affine polar graph** $\mathrm{VO}_{2m}^{\epsilon}(q)$, i.e. vertices can be identified with vectors from $\mathbb{F}_q^{2m}$, and two distinct vertices $u, v \in \Omega$ are adjacent iff $\kappa(u - v) = 0$, where $\kappa : \mathbb{F}_q^{2m} \to \mathbb{F}_q$ is a quadratic form of type ϵ.
$\mathrm{VSz}(q)$	G preserves the **Suzuki–Tits ovoid graph** $\mathrm{VSz}(q)$, i.e. $q = 2^{2e+1}$, vertices can be identified with vectors from $\mathbb{F}_q^4$, and two distinct vertices $u, v \in \Omega$ are adjacent iff the direction of $u - v$ lies in the ovoid $\{(0,0,1,0), (x, y, z, 1) \mid z = xy + x^2 y^{2^{e+1}} + y^{2^{e+1}}\}$, where vectors are written projectively.
$\mathrm{VD}_{5,5}(q)$	G preserves the **affine half-spin graph** $\mathrm{VD}_{5,5}(q)$. The vertices can be identified with vectors from $\mathbb{F}_q^{16}$, and distinct vertices $u, v \in \Omega$ are adjacent iff $u - v$ lies in a certain union of 5-dimensional subspaces. The group G contains a subgroup of the form $\mathbb{F}_q^{16} \rtimes D_5(q)$, where $D_5(q)$ is an orthogonal group.

Exceptions

A finite number of groups is not covered by the main theorem. All of these examples have degree not greater than 3^{12}, and an explicit list can be easily obtained.

Computing 2-closures of these small examples is a nontrivial task, and is a possible future research prospect.

References

[1] M.W. Liebeck, *The affine permutation groups of rank three*, Proceedings of the London Mathematical Society (3) 54, 1987.

[2] A.E. Brouwer, H. Van Maldeghem, *Strongly regular graphs*, preprint.

[3] S.V. Skresanov, *On 2-closures of rank 3 groups*, Ars Mathematica Contemporanea, 2021, doi:10.26493/1855-3974.2450.1dc.

Acknowledgements

The work is supported by Mathematical Center in Akademgorodok under agreement No. 075-15-2019-1613 with the Ministry of Science and Higher Education of the Russian Federation.