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BOUNDARY VALUE PROBLEMS FOR THE LOADED EQUATION WITH INTEGRO-DIFFERENTIAL OPERATOR

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1.1. INTRODUCTION. It is well known that many mathematical models of applied problems require investigations of mixed-type equations. The interest for these equations arises intensively due to both theoretical and practical uses of their applications. At present, the range of problems under consideration for partial differential equations, as well as for mixed types equations has significantly expanded. Along with studying the basic boundary-value problems for such equations, since the seventies, much attention has been paid to the formulation and study of non-local boundary-value problems. This is explained by the fact that many practically important problems related to the dynamics of soil moisture [1], [2], with the diffusion of particles in a turbulent plasma with the cooling of an inhomogeneous curved rod, modeling of the laser radiation process, problems of population dynamics [3] are reduced to the problems with nonlocal boundary conditions. The following loaded equation of the first order [1]

$$\frac{\partial\varphi}{\partial t}+y\frac{\partial\varphi}{\partial x}=(1-\rho)\varphi\int_{\alpha}^{\beta}(y-\eta)\varphi(x,t,\eta)d\eta-\frac{1}{\tau_0}\left[\varphi-G(y)\int_{\alpha}^{\beta}\varphi(x,t,\eta)d\eta\right],$$

is the equation modeling the movement of vehicles on the highway, where $\varphi(x,t,y)$ is the density of the car in the points $x\in[a,b]\subset R$ having a speed, m is the mass and τ_0 is the time of the relaxation, p and G are given values, moreover the function $G=G(y)$ satisfies the condition $\int_{\alpha}^{\beta}G(y)dy=1$.

We can cite by many examples as which emphasized the development of the theory of a loaded equations in the last several decades. As we know [1], [4], mathematical models of nonlocal physics-biological fractal processes represented by loaded integro-differential equations or loaded differential equations, especially with Riemann-Liouville and Caputo operators [5].

On the other hand, another class of researchers confirm the actuality of loaded differential equations, is phenomena in complex evolutionary systems with memory essentially depend on the prehistory of this system and these phenomena is described by following loaded integro-differential equations of elliptic type

$$\Delta_x u(x,t)+\int_0^t\sum_{j=1}^n k_j(t,\tau)\frac{\partial^2 u(x,\tau)}{\partial x_j^2}d\tau=0,$$

parabolic type

$$\frac{\partial u(x,t)}{\partial t}=\Delta_x u(x,t)+\int_0^t k_0(t,\tau)\frac{\partial u(x,\tau)}{\partial\tau}d\tau,$$

where Δ_x- Laplace operators at $x=(x_1,x_2,x_3)$, $k_0(t,\tau),k_j(x,\tau)$ are given real-valued funtions, $j=1,2,\dots,n$.

The following loaded partial differential equation

$$\frac{\partial^2 u}{\partial t^2}-c_0^2\frac{\partial^2 u}{\partial x^2}=mc_0^2\frac{\partial^2}{\partial x^2}\int_{-\infty}^{t}\exp\left(-\frac{t-\eta}{\tau_0}\right)\frac{\partial u(x,\eta)}{\partial\eta}d\eta,$$

are found in the theory of wave propagation in dispersive environments, where $u(x,t)$ is the density, c_0 is the speed, m is the mass and τ_0 is the time of the relaxation.

Example. We consider the following linear loaded [5] integro-differential equation

(1) $$\partial/\partial x\left(u_{xx}-u_{yy}+au_x+bu_y+cu\right)=cD_{0x}^{\alpha}u(x,0),$$

where $cD_{0x}^{\alpha}f(x)$ is a fractional differential operator of Caputo type[6], $0<\alpha<1$, a, b, c are given real parameters.

Let Ω is a characteristic triangle bounded by the segment AB the axis OX and two characteristics

$AC: x+y=0,\quad BC: x-y=1$

of equation (1) for $y<0,\ I=\{(x,y): 0<x<1,\ y=0\}$.

Problem D₁ (Darboux) Find a regular solution $u(x,y)$ of the equation (1) in Ω, which is continuous in $\overline{\Omega}$, has continuous derivatives $u_x(u_y)$, up to $AB\cup AC$, and satisfies the boundary-value conditions

(2) $$u_y(x,0)\big|_{y=0}=\varphi_1(x),\quad 0\le x<1,$$

(3) $$u(x,-x)=\psi_1(x),\quad \frac{\partial u(x,y)}{\partial n}\bigg|_{y=-x}=\psi_2(x),\quad 0\le x\le\frac{1}{2},$$

where $\varphi_1(x),\ \psi_1(x)$ and $\psi_2(x)$ are given functions.

2. THE LOADED EQUATION AND ITS RELATION TO NON-LOCAL PROBLEMS

Theorem 1. Any regular solution of equation (1) is represented in the form

(4) $$u(x,y)=z(x,y)+w(x),$$

where $z(x,y)$ is a solution to the equation

(5) $$\frac{\partial}{\partial x}\left(z_{xx}-z_{yy}+az_x+bz_y+cz\right)=0,$$

$w(x)$ is a solution of the following integro-differential equation

(6) $$w'''(x)+aw''(x)+cw'(x)-cD_{0x}^{\alpha}w(x)=cD_{0x}^{\alpha}z(x,0).$$

Theorem 2. If $\varphi_1(0)=\psi_1(0)$, and $\varphi_1(x)\in C^1[0,1)\cap C^2(0,1)$, $\psi_1(x)\in C^1\left[0,\frac{1}{2}\right]\cap C^3\left(0,\frac{1}{2}\right)$, $\psi_2(x)\in C[0,1/2]\cap C^2(0,1/2)$,

then there exists a unique solution to the problem D₁ in the domain Ω.

Proof. Firstly, by virtue of the representation (4) and in view of (8), the equation (1) and boundary-value conditions (2), (3), are reduced to the form (5),

$$u_y(x,0)=\varphi_1(x),\quad 0\le x<1,$$

$$z(x,-x)=\psi_1(x)-\int_0^x K(x,t)z'(t,0)dt,\quad 0\le x\le\frac{1}{2},$$

$$\frac{\partial z(x,-x)}{\partial n}=\psi_2(x)-\frac{1}{\sqrt{2}}\int_0^x K'(x,t)z'(t,0)dt,\quad 0\le x\le\frac{1}{2},$$

where $K(x,t)$ expressed in terms of $K(x,t)=\int_t^x\left\{K_1(s,t)+\int_t^z R_1(s,z)K_1(z,t)\,dz\right\}ds.$

$$K_1(x,t)=\frac{(x-t)^{2-\alpha}}{\Gamma(1-\alpha)}\int_0^1 v^{-\alpha}(1-v)e^{-\frac{\alpha}{2}(x-t)(1-v)}ds.$$

Since, problem D₁ were reduced to the equivalent non-local problem for a third order equation of mixed type (5), we may conclude that Problem D₁ has a unique solution, as a direct result of the unique solvability of the non-local problem.

3. Conclusion In this work we investigated solvability of boundary-value problems with the continuous gluing conditions for the linear loaded integro-differential equation. The boundary-value problems for loaded integro-differential equations connect to the nonlocal boundary value problems. Wherein boundary-value problems are reduced to the integral equation of Volterra type with the shift and using the successive approximations method was proven an existence of unique solutions of equations.

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